ABOUT THE CONVOLUTION OF DISTRIBUTIONS ON GROUPOIDS

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ABSTRACT. We review the properties of transversality of distributions with respect to submersions. This allows us to construct a convolution product for a large class of distributions on Lie groupoids. We get a unital involutive algebra $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$ enlarging the convolution algebra $C^\infty_c(G, \Omega^{1/2})$ associated with any Lie groupoid G. We prove that G-operators are convolution operators by transversal distributions. We also investigate the microlocal aspects of the convolution product. We give conditions on wave front sets sufficient to compute the convolution product and we show that the wave front set of the convolution product of two distributions is essentially the product of their wave front sets in the symplectic groupoid T^*G of Coste-Dazord-Weinstein. This also leads to a subalgebra $\mathcal{E}'_a(G, \Omega^{1/2})$ of $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$ which contains for instance the algebra of pseudodifferential G-operators and a class of Fourier integral G-operators which will be the central theme of a forthcoming paper.

1. INTRODUCTION

The motivation of this paper is twofold. Firstly, we wish to study the convolution of distributions on a Lie groupoid and its relationship with the action of the so-called G-operators. Secondly, we would like to set up a neat framework in order to investigate in a future work the notions of Lagrangian distributions and Fourier integral operators on a groupoid.

The notion of C^∞ longitudinal family of distributions in the framework of groupoids appears in [16, 18, 23] in order to define right invariant pseudodifferential operators. Also, in the works of Monthubert [14], these families are considered from the point of view of distributions on the whole groupoid, so that the action of the corresponding pseudodifferential operators on C^∞ functions is given by a convolution product. Here, we carry on this idea by exploring the correspondence between C^∞ longitudinal families of distributions and single distributions on the whole underlying manifold of the groupoid and by studying the convolution product of distributions on groupoids. This is achieved at two levels.

The first level is based on the notion of transversality of distributions with respect to a submersion $\pi : M \to B$ [1]. It appears that the space $\mathcal{D}'_\pi(M)$ of such distributions is isomorphic to the space of C^∞ family of distributions in the fibers of π. Also, in the spirit of the Schwartz kernel Theorem suitably stated on the total space of a submersion, the space $\mathcal{D}'_\pi(M)$ coincides with the space of continuous $C^\infty(B)$-linear maps between a suitable subspace $C^\infty_{\mathrm{fc}-\pi}(M)$ of C^∞ functions on M and $C^\infty(B)$. Furthermore, operations such as push-forwards and fibered-products of distributions behave well on transversal distributions and these operations allow to define the convolution product of distributions on groupoids, as soon as these distributions satisfy some transversality assumptions with respect to source or target maps. Distributions on a groupoid which are transversal both to the source and target maps are called bi- transversal and they give rise to an involutive unital algebra $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$ for the convolution product. Then, one has the necessary tools to prove that G-operators on a groupoid are in 1 to 1 correspondence with transversal distributions acting by

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convolution and that bi-transversal distributions are in 1 to 1 correspondence with adjointable G-operators.

The second level is a microlocal refinement of the first one and consists in using the wave front set of distributions. A basic observation, due to Coste, Dazord and Weinstein [4], is that the cotangent manifold T^*G of any Lie groupoid G carries a non trivial structure of symplectic groupoid over the dual of the Lie algebroid A^*G, this structure being intimately related to the multiplication of G and then to the convolution on $C_c^\infty(G, \Omega^{1/2})$. This groupoid combined with the classical calculus of wave front sets developed by Hörmander brings in natural conditions on wave front sets of distributions on a groupoid allowing to define their convolution product and to compute the corresponding wave front set using the law of T^*G. The main consequence of this approach is that the space of compactly supported *admissible* distributions:

$$\mathcal{E}_a'(G, \Omega^{1/2}) = \{u \in \mathcal{E}'(G, \Omega^{1/2}) \; ; \; \mathrm{WF}(u) \cap \ker s_\Gamma = \mathrm{WF}(u) \cap \ker r_\Gamma = \emptyset\},$$

where s_Γ, r_Γ denotes the source and target maps of $T^*G \rightrightarrows A^*G$, is a unital involutive sub-algebra of $(\mathcal{E}_{r,s}'(G, \Omega^{1/2}), *)$ and that

$$\mathrm{WF}(u * v) \subset \mathrm{WF}(u) * \mathrm{WF}(v), \quad \forall u, v \in \mathcal{E}_a'(G, \Omega^{1/2}),$$

where $*$ is the multiplication in the Coste-Dazord-Weinstein groupoid T^*G. We would like to add that the corresponding formula of Hormander for the wave front set of composition of kernels [9, 10] makes the above formula quite predictable. Indeed, given a manifold X, the composition of kernels corresponds to convolution in the pair groupoid $X \times X$ and the composition law that Hörmander defines on $T^*(X \times X)$ to compute wave front sets of composition of kernels is precisely the multiplication map of the Coste-Dazord-Weinstein symplectic groupoid $T^*(X \times X)$.

The distributions belonging to $\mathcal{E}_a'(G, \Omega^{1/2})$ are said to have a bi-transversal wave front set. Actually, this second approach of the convolution product of distributions, based on the groupoid T^*G and Hörmander's techniques, works under assumptions on the wave front sets of distributions weaker than bi-transversality, and we shall briefly develop this point too. However, the algebra $\mathcal{E}_a'(G, \Omega^{1/2})$ is already large enough for the applications that we have in mind. For instance, pseudodifferential G-operators are admissible:

$$\Psi_c(G) \subset \mathcal{E}_a'(G, \Omega^{1/2}).$$

More importantly, if $\Lambda \subset T^*G \setminus 0$ is a homogeneous Lagrangian submanifold of T^*G which is also bi-transversal as a subset of T^*G, then Lagrangian distributions [11] subordinated to Λ are admissible:

$$I^*(G, \Lambda, \Omega^{1/2}) \subset \mathcal{E}_a'(G, \Omega^{1/2})$$

and in particular they give rise to G-operators. This will be the starting point of a second paper.

The present paper is organized as follows. In section 2, we revisit the Schwartz kernel Theorem in the framework of submersions. Then the notion of distributions transversal with respect to a submersion is recalled, we give some examples and we study natural operations available on them. In section 3, we apply the results of section 2 to the case of groupoids. We then define the convolution product of transversal distributions and obtain the unital algebra $\mathcal{E}_{r,s}'(G, \Omega^{1/2})$ of bi-transversal distributions. In section 4, we link the notion of G-operators with the one of transversal distributions and we obtain a 1 to 1 correspondence between the space of adjointable compactly supported G-operators and $\mathcal{E}_{r,s}'(G, \Omega^{1/2})$. In section 5, we use both the Hörmander's results about wave front sets of distributions and the symplectic groupoid structure on T^*G to

identify an important subalgebra of $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$, namely $\mathcal{E}'_a(G, \Omega^{1/2})$ the subspace of distributions with bi-transversal wave front sets, onto which wave front sets behave particulary well with respect to the convolution product.

Finally, we recall in Section 6 the definition of the Coste-Dazord-Weinstein groupoid [4] and add some explanations and comments.

The authors would like to mention that the subject of convolution of transversal distributions is also studied in an independent work by E. Van Erp and R. Yuncken [22].

Aknowledgements

2. Distributions, submersions, transversality

2.1. **Schwartz kernel Theorem for submersions.** To handle distributions on groupoids, it is useful to study distributions in the total space of a submersion. The notion of transversality we shall recall is borrowed from [1] and it extends the condition of semi-regularity given in [21, p.532].

For any manifold M and real number α, the bundle of α-densities is denoted by Ω^α_M. The space $\mathcal{D}'(M, \Omega^\alpha_M)$ (resp. $\mathcal{E}'(M, \Omega^\alpha_M)$) is the topological dual of the space $C^\infty_c(M, \Omega^{1-\alpha}_M)$ (resp. $C^\infty(M, \Omega^{1-\alpha}_M)$). With the convention chosen, we have canonical topological embeddings

$$C^\infty(M, \Omega^\alpha) \hookrightarrow \mathcal{D}'(M, \Omega^\alpha)$$

and we abbreviate $\mathcal{D}'(M) = \mathcal{D}'(M, \Omega^0_M)$, $\Omega_M = \Omega^1_M$.

Distributions spaces are provided with the strong topology. The space of continuous linear maps between two locally convex vector spaces E, F is denoted by $\mathcal{L}(E, F)$ and provided with the topology of uniform convergence on bounded subsets. If E, F are modules over an algebra A, the subspace of continuous A-linear maps between E and F is denoted by $\mathcal{L}_A(E, F)$ and considered as a topological subspace of $\mathcal{L}(E, F)$.

We are going to reformulate the Schwartz kernel Theorem for distributions in the total space of a submersion $\pi : M \longrightarrow B$ between C^∞-manifolds. To do this, we begin with the product case $\pi = \mathrm{pr}_1 : X \times Y \longrightarrow X$ where $X \subset \mathbb{R}^{n_X}$ and $Y \subset \mathbb{R}^{n_Y}$ denote open subsets.

The Schwartz kernel Theorem then asserts that the map

$$(1) \qquad \mathcal{D}'(X \times Y) \ni u \longmapsto \left(f \longmapsto u_f(x) = \int_Y u(x, y) f(y) dy \right) \in \mathcal{L}(C^\infty_c(Y), \mathcal{D}'(X))$$

where the integral is understood in the distribution sense, is a topological isomorphism. This can be seen as a push-forward operation along the fibers of π and to state this for an arbitrary submersion $\pi : M \longrightarrow B$ between C^∞-manifolds, we introduce the space

$$(2) \qquad C^\infty_{\mathrm{fc}-\pi}(M) = \{ f \in C^\infty(M) \; ; \; \pi : \mathrm{supp}(f) \to B \text{ is proper } \}.$$

This is the LF-space associated with the sequence of Frechet spaces $\{ f \in C^\infty(M) \; ; \; \mathrm{supp}(f) \subset F_n \} = C^\infty_0(F_n)$ where (F_n) is an exhausting sequence of closed subsets of M such that $\pi : F_n \to B$ is proper.

The injections $C^\infty_c \hookrightarrow C^\infty_{\mathrm{fc}-\pi} \hookrightarrow C^\infty$ are continuous and C^∞_c is dense in $C^\infty_{\mathrm{fc}-\pi}$. When B is compact, we have $C^\infty_{\mathrm{fc}-\pi} = C^\infty_c$. Vector bundles over M can be added and we do not repeat the definitions. Then for any $f \in C^\infty_{\mathrm{fc}-\pi}(M, \Omega_M)$, one can associate a distribution $\pi_*(uf)$ on B defined

for any $g \in C_c^\infty(B, \Omega_B)$ by

$$(3) \qquad\qquad \langle \pi_*(uf), g \rangle = \langle uf, g \circ \pi \rangle = \langle u, f.g \circ \pi \rangle.$$

One can view naturally $C_{\text{fc}-\pi}^\infty(M, \Omega_M)$ as a $C^\infty(B)$-module by using π : for $f \in C_{\text{fc}-\pi}^\infty(M, \Omega_M)$ and $g \in C_c^\infty(B, \Omega_B)$, one defines $f.g$ on M by $f.g(m) = f(m)g(\pi(m))$ and the condition on the support is obvious.

We have

Theorem 1 (Schwartz kernel Theorem for submersions)**.** *The map*

$$\pi_* : \mathcal{D}'(M) \longrightarrow \mathcal{L}_{C^\infty(B)}(C_{\text{fc}-\pi}^\infty(M, \Omega_M), \mathcal{D}'(B, \Omega_B))$$

$$u \longmapsto \pi_*(u\cdot)$$

is a topological isomorphism.

Proof. Let C be a bounded subset of $C_{\text{fc}-\pi}^\infty(M, \Omega_M)$ and D be a bounded subset of $C_c^\infty(B)$. Then $C.D = \{f.g \; ; \; f \in C, g \in D\}$ is a bounded subset of $C_c^\infty(M, \Omega_M)$. The continuity of π_* follows. Conversely, we define $I : \mathcal{L}_{C^\infty(B)}(C_{\text{fc}-\pi}^\infty(M, \Omega_M), \mathcal{D}'(B, \Omega_B)) \to \mathcal{D}'(M)$ by

$$(4) \qquad \langle I(T), f \rangle = \langle T(f), \psi \rangle \quad f \in C_c^\infty(M, \Omega_M), \; \psi \in C_c^\infty(B), \; f\psi = f.$$

The definition of $I(T)$ as a linear form on $C_c^\infty(M, \Omega_M)$ is consistant for T being $C^\infty(B)$-linear, the left hand side does not depend on the choice of ψ such that $f\psi = f$. Moreover, if E is a bounded subset of $C_c^\infty(M, \Omega_M)$ then there exists a compact subset $K \subset M$ such that $f \in E$ implies $\text{supp}(f) \subset K$. Fixing $\psi \in C_c^\infty(B)$ such that $\psi = 1$ onto K yields that $I(T)$ is a distribution for any T and the continuity of the map I. The relations $\pi_* \circ I = \text{Id}$ and $I \circ \pi_* = \text{Id}$ are obvious. $\square$

Remark 2. Playing with supports, we also get

$$\mathcal{E}'(M) \simeq \mathcal{L}_{C^\infty(B)}(C^\infty(M, \Omega_M), \mathcal{E}'(B, \Omega_B)) \text{ and } \mathcal{D}'_{\text{fc}-\pi}(M) \simeq \mathcal{L}_{C^\infty(B)}(C^\infty(M, \Omega_M), \mathcal{D}'(B, \Omega_B)).$$

Here $\mathcal{D}'_{\text{fc}-\pi}(M)$ is the topological dual of the LF-space $\{f \in C^\infty(M, \Omega_M) \; ; \; \pi(\text{supp}(f)) \text{ is compact}\}$.

2.2. **Transversal distributions.**

Definition 1. *([1] Androulidakis-Skandalis). A distribution $u \in \mathcal{D}'(M)$ is transversal to π if $\pi_*(u.f) \in C^\infty(B, \Omega_B)$ for any $f \in C_{\text{fc}-\pi}^\infty(M, \Omega_M)$. We note $\mathcal{D}'_\pi(M)$ the space of π-transversal distributions. We also set*

$$\mathcal{E}'_\pi(M) = \mathcal{D}'_\pi(M) \cap \mathcal{E}'(M) \text{ and } \mathcal{P}'_\pi(M) = \mathcal{D}'_\pi(M) \cap \mathcal{D}'_{\text{fc}-\pi}(M)$$

Observe that if u is π-transversal, it follows from the closed graph theorem for LF-spaces [12, Cor 1.2.20, p. 22] that $\pi_*(u\cdot) \in \mathcal{L}(C_{\text{fc}-\pi}^\infty(M, \Omega_M), C^\infty(B, \Omega_B))$. This gives

Proposition 3. *Denoting by π_* the isomorphism in Theorem 1, we have*

$$(5) \qquad \pi_*(\mathcal{D}'_\pi(M)) = \mathcal{L}_{C^\infty(B)}(C_{\text{fc}-\pi}^\infty(M, \Omega_M), C^\infty(B, \Omega_B)).$$

Remark 4. *Similarly,*

$$(6) \qquad \pi_*(\mathcal{E}'_\pi(M)) = \mathcal{L}_{C^\infty(B)}(C^\infty(M, \Omega_M), C_c^\infty(B, \Omega_B)),$$

$$(7) \qquad \pi_*(\mathcal{P}'_\pi(M)) = \mathcal{L}_{C^\infty(B)}(C^\infty(M, \Omega_M), C^\infty(B, \Omega_B)).$$

In all cases, the inverse of the map π_* is given by

(8) $$\langle \pi_*^{-1}(T), f \rangle = \int_B T(f) , \quad f \in C_c^\infty(M, \Omega_M).$$

When $\pi : X \times Y \to X$, $(x, y) \to x$, the π-transversal distributions are exaclty the distributions semi-regular with respect to x, in the former terminology of [21, p.532].

Actually, transversal distributions are nothing else but C^∞ families of distributions in the fibers of π. In the product case $\pi : X \times Y \to X, (x, y) \mapsto x$, we are talking about the space $C^\infty(X, \mathcal{D}'(Y))$ C^∞ functions on X taking values in the topological vector space $\mathcal{D}'(Y)$ [21]. Since $\mathcal{D}'(Y)$ is a Montel space, the classical argument using Banach-Steinhaus Theorem shows the useful equivalence

(9) $$u_n \longrightarrow u \text{ in } C^\infty(X, \mathcal{D}'(Y)) \iff \forall f \in C_c^\infty(Y), \ \langle u_n, f \rangle \longrightarrow \langle u, f \rangle \text{ in } C^\infty(X).$$

This space is generalized as follows for general submersions.

Definition 2. *A family $u = (u_x)_{x \in B}$ of distributions in the fibers of π is C^∞ if for any local trivialization of π*

$$U \subset M, \ X \subset B, \ \kappa : U \xrightarrow{\simeq} X \times Y, \ \pi|_U = \pi_X \circ \kappa,$$

we have $\kappa_*(u|_U) \in C^\infty(X, \mathcal{D}'(Y))$. The space of C^∞ families is noted $C_\pi^\infty(B, \mathcal{D}'(M))$. The spaces $C_{\pi,\mathrm{cpct}}^\infty(B, \mathcal{E}'(M))$ and $C_\pi^\infty(B, \mathcal{D}'_{\mathrm{fc}-\pi}(M))$ are defined accordingly.

Using a covering of M by local trivializations and a partition of unity, we use the topology of $C^\infty(X, \mathcal{D}'(Y))$ to build on $C_\pi^\infty(B, \mathcal{D}'(M))$ a complete Hausdorff locally convex vector space structure. Concretely, this topology is given by the semi-norms generated by the following ones expressed in local coordinates

(10) $$p_{k,\mathcal{B},K}(u) = \sup_{x \in K, g \in \mathcal{B}} \sum_{|\alpha| \le k} |\langle \partial_x^\alpha u(x), g(x, \cdot) \rangle|$$

where k is any integer, K any compact subset included in a local chart of B and $\mathcal{B}$ any bounded subset of $C_{\mathrm{fc}-\pi}^\infty(M, \Omega_M)$. Also, (9) becomes

(11) $$u_n \longrightarrow u \text{ in } C_\pi^\infty(B, \mathcal{D}'(M)) \iff \forall f \in C_{\mathrm{fc}-\pi}^\infty(M, \Omega_M), \ \langle u_n, f \rangle \longrightarrow \langle u, f \rangle \text{ in } C^\infty(B, \Omega_B).$$

Then

Proposition 5. *Using on $\mathcal{D}'_\pi(M)$ the topology given by (5), the map*

(12) $$C_\pi^\infty(B, \mathcal{D}'(M)) \xrightarrow{J} \mathcal{D}'_\pi(M)$$

$$u \longmapsto \left(f \mapsto \int_B \langle u_x, f(x, \cdot) \rangle \right)$$

is a topological isomorphism.

Proof. Using the identification $\mathcal{D}'_\pi(M) \simeq \pi_*(\mathcal{D}'_\pi(M))$, the map J is given by

$$J(u)(f)(x) = \langle u_x, f(x, \cdot) \rangle, \ u \in C_\pi^\infty(B, \mathcal{D}'(M)), f \in C_{\mathrm{fc}-\pi}^\infty(M, \Omega_M), \ x \in B.$$

Conversely, let us define $\pi_*(\mathcal{D}'_\pi(M)) \xrightarrow{E} C_\pi^\infty(B, \mathcal{D}'(M))$ by

(13) $$\langle E(T)_x, f \rangle = T(\widetilde{f})(x)$$

where $f \in C_c^\infty(\pi^{-1}(x), \Omega_M|_{\pi^{-1}(x)})$ and $\widetilde{f} \in C_c^\infty(M, \Omega_M)$ is any C^∞ extension of f. It is easy to check that $E = J^{-1}$ and that the topology given by the semi-norms (10) on $C_\pi^\infty(B, \mathcal{D}'(M))$ coincides with the one given by uniform convergence on bounded subsets for the space $\pi_*(\mathcal{D}'_\pi(M))$ through the bijection J. $\square$

Remark 6. *We similarly get*

$$C^\infty_{\pi,\mathrm{cpct}}(B, \mathcal{E}'(M)) \simeq \mathcal{E}'_\pi(M) \text{ and } C^\infty_\pi(B, \mathcal{D}'_{\mathrm{fc}-\pi}(M)) \simeq \mathcal{P}'_\pi(M).$$

If vector bundles E over M and F over B are given, we obtain canonical embeddings

(14) $\mathcal{D}'_\pi(M, E) \hookrightarrow \mathcal{D}'_\pi(M, E \otimes \mathrm{End}(\pi^* F)) \simeq \mathcal{L}_{C^\infty(B)}(C^\infty_{\mathrm{fc}-\pi}(M, \Omega_M \otimes E^* \otimes \pi^* F), C^\infty(B, \Omega_B \otimes F))$

and

(15) $C^\infty_\pi(B, \mathcal{D}'(M, E)) \hookrightarrow C^\infty_\pi(B, \mathcal{D}'(M, E \otimes \mathrm{End}(\pi^* F))) \simeq \mathcal{D}'_\pi(M, E \otimes \mathrm{End}(\pi^* F)).$

2.3. **Examples of transversal distributions.**

Obviously, if $\pi : M \to M$ is the identity map then $\mathcal{D}'_\pi(M) = C^\infty(M)$ and if π maps M to a point then $\mathcal{D}'_\pi(M) = \mathcal{D}'(M)$.

The wave front set ([10, Chapter 8]) is a powerful tool to analyse the singularities of a distribution. It can be thought of as the set of directed points in $T^* M \setminus 0$, around which the Fourier transform is not rapidly decreasing. Using wave front set is a convenient way to check the transversality of distributions with respect to a given submersion $\pi : M \to B$, and it thus gives access to more interesting examples. Indeed,

Proposition 7. *Let $W \subset T^* M \setminus 0$ be a closed cone and $\mathcal{D}'_W(M) = \{u \in \mathcal{D}'(M) \; ; \; \mathrm{WF}(u) \subset W\}$. If $W \cap (\ker d\pi)^\perp = \emptyset$, then*

$$\mathcal{D}'_W(M) \subset \mathcal{D}'_\pi(M).$$

Proof. We apply the formula (3.6) p. 328 of [8]:

$$\mathrm{WF}(\pi_*(u.f)) \subset (d\pi)_*(\mathrm{WF}(u.f)) \subset (d\pi)_*(\mathrm{WF}(u)) = \{(x, \xi) \; ; \; x = \pi(m) \; , (m, {}^t d\pi_m(\xi)) \in \mathrm{WF}(u)\}.$$

Since $(\ker d\pi)^\perp = \{(m, \zeta) \; ; \; \zeta \in \mathrm{Im}({}^t d\pi_m)\}$, we obtain $\mathrm{WF}(\pi_*(u.f)) = \emptyset$, and thus $\pi_*(u.f)$ is smooth. $\qquad\square$

For instance, consider a section of π, that is a submanifold $X \subset M$ such that $\pi : X \to B$ is a diffeomorphism onto an open subset of B. Let $\omega \in \Omega(X)$ be any C^∞ density and define $l_\omega \in \mathcal{D}'(M, \Omega_M)$ by

(16) $$\langle l_\omega, f \rangle = \int_X f\omega.$$

Then $l_\omega \in \mathcal{D}'_\pi(M, \Omega_M)$, for $\mathrm{WF}(l_\omega) \subset N^*(X)$ (see [10, Example 8.2.5]) and $N^*(X) \cap (\ker d\pi)^\perp = X \times \{0\}$. Alternatively, it is easy to check that $\pi_*(l_\omega.f)$ is given by the C^∞ density $\pi_*(\omega f|_X)$. Of course, for any differential operator P on M, we still have $P l_\omega \in \mathcal{D}'_\pi(M, \Omega_M)$, for $\mathrm{WF}(Pu) \subset \mathrm{WF}(u)$ for any distribution u. Actually, this gives all instances of transversal distributions supported within a section. Indeed, let $u \in \mathcal{E}'_\pi(M, \Omega_M)$ such that $\mathrm{supp}(u) \subset X$. It is no restriction to work in a local trivialization, that is to assume $\pi : M = X \times \mathbb{R}^n \to X$, $(x, y) \mapsto x$ and identify $X \simeq X \times \{0\}$. By [10, Theorem 2.3.5], we have

(17) $$\langle u, \phi \rangle = \sum_{|\alpha| \le k} \langle u_\alpha, (\partial_y^\alpha \phi)(\cdot, 0) \rangle, \qquad \forall \phi \in C^\infty_c(X \times \mathbb{R}^n)$$

where k is the order of u and $u_\alpha \in \mathcal{D}'(X)$ has order $k - |\alpha|$. It follows that

(18) $$C^\infty(X) \ni \pi_*(fu) = \sum_{|\alpha| \le k} (\partial_y^\alpha f)(\cdot, 0).u_\alpha, \quad \forall f \in C^\infty(X \times \mathbb{R}^n).$$

Selecting $f = y^\alpha$ shows that u_α is C^∞. We have proved

Proposition 8. *Let $u \in \mathcal{E}'(M, \Omega_M)$ such that $\mathrm{supp}(u) \subset X$, X being a section of π. Then $u \in \mathcal{E}'_\pi(M, \Omega_M)$ if and only if u is a finite sum of distributions obtained by differentiation along the fibers of π of distributions of the kind (16).*

Remark 9. *$u \in \mathcal{D}'_\pi(M)$ does not imply $\mathrm{WF}(u) \cap \ker d\pi^\perp = \emptyset$. Indeed, consider $\pi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (x, y) \mapsto x$ and define $u \in C^\infty(\mathbb{R}, \mathcal{D}'(\mathbb{R}))$ by*

$$\langle u, \phi \rangle(x) = \sqrt{2\pi} \int \chi(\eta) |\eta| e^{-\eta^2 x^2/2} \hat{\phi}(-\eta) d\eta \tag{19}$$

where χ is C^∞, $\chi(\eta) = 1$ if $|\eta| \geq 1$ and $\chi(\eta) = 0$ if $|\eta| \leq 1/2$. Since $\hat{u}(\xi, \eta) = \chi(\eta) e^{-\xi^2/(2\eta^2)}$ we conclude $\mathrm{WF}(u) \cap (\ker d\pi)^\perp \neq \emptyset$ ([10, Section 8.1]).

It is not obvious to us how to characterize transversal distributions whose wave front set avoids $(\ker d\pi)^\perp$. We give in the following lemma a sufficient condition.

Lemma 10. *Let $v \in \mathcal{D}'_{\pi_X}(X \times Y)$ and assume that there exists constants $d \in \mathbb{N}$ and $\delta \in [0, 1)$ such that for any compact subset K of Y and multi-index $\beta \in \mathbb{N}^{n_X}$, one can find a constant $C_{K\beta}$ such that*

$$|\langle \partial^\beta v_x, f \rangle| \leq C_{K\beta} \|f\|_{K, d+\delta|\beta|}, \quad \forall f \in C_c^\infty(Y), \ x \in X. \tag{20}$$

Here, we have set $\|f\|_{K, d+\delta|\beta|} = \sum_{|\alpha| \leq d+\delta|\beta|} \sup_K |\partial^\alpha f|$. Then we have $\mathrm{WF}(v) \subset (\ker d\pi_Y)^\perp$. In particular, $\mathrm{WF}(v) \cap (\ker d\pi_X)^\perp = \emptyset$.

Remark 11. Distributions in Proposition 8 satisfy the assumption of the lemma with $\delta = 0$.

Proof of the lemma. Let us fix $(x_0, y_0, \xi_0, \eta_0) \notin (\ker d\pi_Y)^\perp$, that is, $\xi_0 \neq 0$ and assume that $|(\xi_0, \eta_0)| = 1$. We work below in a conic neighborhood Γ of $(x_0, y_0, \xi_0, \eta_0)$ such that for all $(x, y, \xi, \eta) \in \Gamma$ with $|(\xi, \eta)| = 1$, we have $|\xi_j| \geq |\xi_{0j}|/2$ for some fixed j such that $\xi_{0j} \neq 0$.

Let $(x, y, \xi, \eta) \in \Gamma$ be such that $|(\xi, \eta)| = 1$ and $\varphi(x, y)$ be supported in a compact neighborhood $K \times L$ of (x_0, y_0) in $X \times Y$. Denoting $\varphi_x = \varphi(x, \cdot)$, we have for any $N > 0$

$$|\langle v, \varphi e^{-it\langle(\cdot, \cdot), (\xi, \eta)\rangle} \rangle| = |\int \langle v_x, \varphi_x e^{-it\langle(x, \cdot), (\xi, \eta)\rangle} \rangle dx| = |\int \widehat{\varphi_x v_x}(t\eta) e^{-it\langle x, \xi \rangle} dx|$$

$$\leq C \cdot \left(\sum_{|\alpha| \leq N} \sup_{x \in L} |\partial_x^\alpha \widehat{\varphi_x v_x}(t\eta)| |\xi|^{|\alpha| - 2N} \right) t^{-N} \text{ by } [10, \text{ Theorem 7.7.1}]. \tag{21}$$

Moreover, since $v : x \mapsto v_x$ is C^∞, we have

$$\partial_{x_j}^N \widehat{\varphi_x v_x}(t\eta) = \partial_{x_j}^N \langle \varphi_x v_x, e^{-it\langle \cdot, \eta \rangle} \rangle = \langle \partial_{x_j}^N \varphi_x v_x, e^{-it\langle \cdot, \eta \rangle} \rangle = \widehat{\partial_{x_j}^N \varphi_x v_x}(t\eta).$$

We note $K_\epsilon = \{y + z; \ y \in K, |z| < \epsilon\}$ for any $\epsilon > 0$ and let $\chi_\epsilon \in C_c^\infty(K_\epsilon)$ be such that $\chi_\epsilon = 1$ on $K_{\epsilon/2}$. If $H(\eta)$ denotes the supporting function of K [10, 4.3.1], we get using the assumption (20) and the proof of the Paley-Wiener-Schwartz Theorem in [10, 7.3.1]

$$|\widehat{\partial_{x_j}^N \varphi_x v_x}(\eta)| = |\partial_{x_j}^N \varphi_x v_x(\chi_\epsilon e^{-i\langle \cdot, \eta \rangle})| \leq C_{K_\epsilon N} \sum_{|\beta| \leq d + \delta N} \sup |\partial^\beta (\chi_\epsilon e^{-i\langle \cdot, \eta \rangle})|$$

$$\leq C_{K_\epsilon N} \cdot C \cdot e^{H(0)} \cdot \sum_{|\beta| \leq d + \delta N} \epsilon^{-\beta} (1 + |\eta|)^{d + \delta N - |\beta|}.$$

With $\epsilon = 1/(1 + |\eta|)$ and using the inequalities $C_{K_\epsilon N} \leq C_{K_{\epsilon'} N}$ if $\epsilon < \epsilon'$, we obtain

$$|\widehat{\partial_{x_j}^N \varphi_x v_x}(\eta)| \leq C_{K_1 N} \cdot C \cdot (1 + |\eta|)^{d + \delta N} \leq C'_{KN} (1 + |\eta|)^{d + \delta N}. \tag{22}$$

Using uniform estimates $|\xi| \geq c_1 > 0$ and $(1 + |t\eta|) \leq c_2 t$ for $(\xi, \eta) \in \Gamma$, $|(\xi, \eta)| = 1$ and the estimate (22) applied to (21), we get

$$|\langle v, \varphi e^{-it\langle (\cdot, \cdot), (\xi, \eta) \rangle} \rangle| \quad \leq \quad C.t^{d + (\delta - 1)N}.$$

since $\delta - 1 < 0$, we conclude that $(x_0, y_0, \xi_0, \eta_0) \notin \mathrm{WF}(v)$. $\qquad\square$

2.4. Operations on transversal distributions.

One has obviously, following the definitions.

Proposition 12. *Let E be a C^∞ vector bundle over M, C a submanifold of B and $\pi_C : \pi^{-1}(C) \to C$ the restriction of the submersion $\pi : M \to B$. The restriction of distributions*

$$\mathrm{Rest}_C : \mathcal{D}'_\pi(M, E) \longrightarrow \mathcal{D}'_{\pi_C}(\pi^{-1}(C), E|_{\pi^{-1}(C)})$$

is well defined and continuous.

Proposition 13. *Let $\rho : Z \to M$ and $\pi : M \to B$ be surjective submersions. Let E be a C^∞ vector bundle over M. The pull back of distributions restricts to a continuous map*

$$\rho^* : \mathcal{D}'_\pi(M, E) \longrightarrow \mathcal{D}'_{\pi \circ \rho}(Z, \rho^* E).$$

Proof. Since ρ is a submersion, the map

$$\rho_* : \quad C^\infty_{\mathrm{fc} - \pi \circ \rho}(Z, \Omega_Z \otimes \rho^* E^*) \quad \longrightarrow \quad C^\infty_{\mathrm{fc} - \pi}(M, \Omega_M \otimes E^*)$$
$$f \qquad\qquad \longmapsto \qquad (m \mapsto \int_{\rho^{-1}(m)} f)$$

is well defined and continuous. Since $\rho^*(u) = u \circ \rho_*$, the proposition follows. $\qquad\square$

Let $\pi_i : M_i \longrightarrow B$, $i = 1, 2$ be two submersions and define

$$\pi : M_1 \underset{\pi}{\times} M_2 = \{(m_1, m_2) \in M_1 \times M_2 \; ; \; \pi_1(m_1) = \pi_2(m_2)\} \ni (m_1, m_2) \longmapsto \pi_1(m_1).$$

Writing $\mathrm{pr}_i(m_1, m_2) = m_i$, we get a commutative square of submersions

$$(23) \qquad\qquad \begin{array}{ccc} M_1 \underset{\pi}{\times} M_2 & \overset{\mathrm{pr}_2}{\longrightarrow} & M_2 \\ {\scriptstyle \mathrm{pr}_1} \downarrow & & \downarrow {\scriptstyle \pi_2} \\ M_1 & \overset{\pi_1}{\longrightarrow} & B \end{array}$$

Proposition 14. *The pull-back $\mathrm{pr}_1^* : \mathcal{D}'(M_1) \longrightarrow \mathcal{D}'(M_1 \underset{\pi}{\times} M_2)$ restricts to a continuous map*

$$(24) \qquad\qquad \mathrm{pr}_1^* : \mathcal{D}'_{\pi_1}(M_1) \longrightarrow \mathcal{D}'_{\mathrm{pr}_2}(M_1 \underset{\pi}{\times} M_2).$$

Proof. We identify transversal distributions with C^∞ families and we can work locally, that is we assume that $\pi_j : X \times Y_j \to X$, with X, Y_1, Y_2 open subsets in euclidean spaces. If $u \in C^\infty(X, \mathcal{D}'(Y_1))$ then $\mathrm{pr}_1^*(u)$ is given by the family

$$X \times Y_2 \ni (x, y_2) \longmapsto u_x \in \mathcal{D}'(Y_1).$$

The statement follows.

$\qquad\square$

Remark 15. (1) The assertion of the previous proposition holds for commutative square of surjective submersions

(25)
$$\begin{array}{ccc} M & \xrightarrow{\mathrm{pr}_2} & M_2 \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{\pi_1} & B \end{array}$$

such that any point of M, $\ker d\pi_1 \circ \mathrm{pr}_1 = \ker d\,\mathrm{pr}_1 + \ker d\,\mathrm{pr}_2$ or, equivalently, such that $\mathrm{pr}_1 : \mathrm{pr}_2^{-1}(m_2) \to \pi_1^{-1}(b)$, $b = \pi_2(m_2)$, is a submersion for any $m_2 \in M_2$.

(2) One can also prove in the same way that the restriction $\mathrm{pr}_1^* : \mathcal{P}'_{\pi_1}(M_1) \longrightarrow \mathcal{P}'_{\mathrm{pr}_2}(M)$ is continuous, observing that pr_1^* restricts to $\mathcal{D}'_{\mathrm{fc}-\pi_1}(X \times Y_1) \longrightarrow \mathcal{D}'_{\mathrm{fc}-\mathrm{pr}_2}(X \times Y_1 \times Y_2)$.

Then one can, as above, extend this result to more general commutative squares, provided the additional condition that the map $M \to M_1 \underset{\pi}{\times} M_2, m \mapsto (\mathrm{pr}_1(m), \mathrm{pr}_2(m))$ has compact fibers.

When a finite set $\mathcal{I}$ of submersions is given on M, we introduce

(26)
$$\mathcal{D}'_{\mathcal{I}}(M) = \bigcap_{\rho \in \mathcal{I}} \mathcal{D}'_\rho(M) \subset \mathcal{D}'(M).$$

The space $\mathcal{D}'_{\mathcal{I}}(M, E)$ is given the topology generated by the union of the topologies induced by each $\mathcal{D}'_\rho(M)$, $\rho \in \mathcal{I}$. We adopt similar convention for the spaces $\mathcal{E}'_{\mathcal{I}}(M)$ and $\mathcal{D}'_{c-\mathcal{I}}(M)$. The previous proposition is now used to define fibered product of distributions.

Proposition 16. *We keep the setting of Proposition 14 and we consider extra submersions $\rho : M_1 \longrightarrow A$, $\sigma : M_2 \longrightarrow C$.*

$$\begin{array}{ccccc} M_1 \underset{\pi}{\times} M_2 & \xrightarrow{\mathrm{pr}_2} & M_2 & \xrightarrow{\sigma} & C \\ \downarrow{\scriptstyle \mathrm{pr}_1} & \searrow{\scriptstyle \pi} & \downarrow{\scriptstyle \pi_2} & & \\ M_1 & \xrightarrow{\pi_1} & B & & \\ \downarrow{\scriptstyle \rho} & & & & \\ A & & & & \end{array}$$

The fibered product of C^∞ functions $(f_1, f_2) \longmapsto f_1 \otimes f_2|_{M_1 \underset{\pi}{\times} M_2}$ extends uniquely to separately continuous bilinear maps

(27)
$$\begin{array}{ccc} \mathcal{D}'_{\pi_1}(M_1) \times \mathcal{D}'_\sigma(M_2) & \longrightarrow & \mathcal{D}'_{\sigma \circ \mathrm{pr}_2}(M) \\ (u_1, u_2) & \longmapsto & u_1 \underset{\pi_1}{\times} u_2 \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{D}'_\rho(M_1) \times \mathcal{D}'_{\pi_2}(M_2) & \longrightarrow & \mathcal{D}'_{\rho \circ \mathrm{pr}_1}(M) \\ (u_1, u_2) & \longmapsto & u_1 \underset{\pi_2}{\times} u_2 \end{array}$$

If $u_j \in \mathcal{D}'_{\pi_j}(M_j)$, $j = 1, 2$ then the equality

(28)
$$u_1 \underset{\pi_1}{\times} u_2 = u_1 \underset{\pi_2}{\times} u_2$$

holds and both previous maps restrict to a separately continuous bilinear map

(29)
$$\begin{array}{ccc} \mathcal{D}'_{\rho, \pi_1}(M_1) \times \mathcal{D}'_{\pi_2, \sigma}(M_2) & \longrightarrow & \mathcal{D}'_{\rho \circ \mathrm{pr}_1, \pi, \sigma \circ \mathrm{pr}_2}(M) \\ (u_1, u_2) & \longmapsto & u_1 \underset{\pi_1}{\times} u_2. \end{array}$$

Remark 17. The above map in (27) restricts to a separately continuous map

(30)
$$\mathcal{P}'_{\pi_1}(M_1) \times \mathcal{D}'_\sigma(M_2) \longrightarrow \mathcal{D}'_{\sigma \circ \mathrm{pr}_2}(M) \cap \mathcal{D}'_{\mathrm{fc}-\mathrm{pr}_2}(M).$$

There are analogous statements for the fibered product over π_2 and the conditions on supports can be interchanged.

Proof. Let $u_1 \in \mathcal{D}'_{\pi_1}(M_1)$ and $u_2 \in \mathcal{D}'(M_2)$. As a distribution on M, $u_1 \underset{\pi_1}{\times} u_2$ is defined by

$$(31) \qquad \langle u_1 \underset{\pi_1}{\times} u_2, f \rangle = \int_{M_2} \left(\int_{M_{1\pi_2(m_2)}} u_{1\pi_2(m_2)}(m_1) f(m_1, m_2) \right) u_2(m_2)$$

where the integrals are taken in the distribution sense. This coincides with the fibered product of functions when u_1, u_2 are C^∞. Proposition 14 says that

$$(32) \qquad T : u \in \mathcal{D}'_{\pi_1}(M_1) \longmapsto \mathrm{pr}_1^*(u) \in \mathcal{L}(C^\infty_{\mathrm{fc}-\mathrm{pr}_2}(M, \Omega_M), C^\infty(M_2, \Omega_{M_2}))$$

is well defined and continuous. Observing that the inclusion $C^\infty_{\mathrm{fc}-\sigma \circ \mathrm{pr}_2}(M, \Omega_M) \subset C^\infty_{\mathrm{fc}-\mathrm{pr}_2}(M, \Omega_M)$ is continuous, we get the continuity of the mapping:

$$(33) \qquad T : u \in \mathcal{D}'_{\pi_1}(M_1) \longmapsto \mathrm{pr}_1^*(u) \in \mathcal{L}(C^\infty_{\mathrm{fc}-\sigma \circ \mathrm{pr}_2}(M, \Omega_M), C^\infty(M_2, \Omega_{M_2})).$$

From $\mathrm{pr}_1^*(u)(C^\infty_{\mathrm{fc}-\sigma \circ \mathrm{pr}_2}(M, \Omega_M)) \subset C^\infty_{\mathrm{fc}-\sigma}(M_2, \Omega_{M_2}))$, we get from the closed graph Theorem again that the restricted map

$$\mathrm{pr}_1^*(u) : C^\infty_{\mathrm{fc}-\sigma \circ \mathrm{pr}_2}(M, \Omega_M) \longrightarrow C^\infty_{\mathrm{fc}-\sigma}(M_2, \Omega_{M_2})$$

is continuous, and we outline the proof of continuity of the mapping:

$$T : u \in \mathcal{D}'_{\pi_1}(M_1) \longmapsto \mathrm{pr}_1^*(u) \in \mathcal{L}(C^\infty_{\mathrm{fc}-\sigma \circ \mathrm{pr}_2}(M, \Omega_M), C^\infty_{\mathrm{fc}-\sigma}(M_2, \Omega_{M_2})).$$

By definition of the topology of uniform convergence on bounded subsets, we have to estimate

$$(34) \qquad p_\mathcal{B}(T(u)) = \sup_{f \in \mathcal{B}} p(T(u)(f))$$

for any semi-norm p defining the topology of $C^\infty_{\mathrm{fc}-\sigma}(M_2, \Omega_{M_2})$ and any bounded subset $\mathcal{B} \subset C^\infty_{\mathrm{fc}-\sigma \circ \mathrm{pr}_2}(M, \Omega_M)$. But for any such bounded subset $\mathcal{B}$, there exists $\Omega \subset M$ such that $\sigma \circ \mathrm{pr}_2 : \Omega \to C$ is proper and $f \in \mathcal{B} \Rightarrow \mathrm{supp}(f) \subset \Omega$. We then have $\mathrm{supp}(T(u)(f)) \subset \mathrm{pr}_2(\Omega)$ for any $\mathcal{B}$. Since $\sigma : \mathrm{pr}_2(\Omega) \to C$ is proper, p is also a continuous semi-norm on $C^\infty_0(\mathrm{pr}_2(\Omega), \Omega_{M_2})$ hence we can replace in (34) p by a semi-norm of $C^\infty_0(\mathrm{pr}_2(\Omega), \Omega_{M_2})$ or equally by a semi-norm of $C^\infty(M_2, \Omega_{M_2})$ and the result now follows from the continuity of (33). Then

$$(35) \qquad u_1 \underset{\pi_1}{\times} u_2 = u_2 \circ \mathrm{pr}_1^*(u_1) \in \mathcal{L}_{C^\infty(C)}(C^\infty_{\mathrm{fc}-\sigma \circ \mathrm{pr}_2}(M, \Omega_M), C^\infty(C, \Omega_C))$$

is continuous in u_1 and u_2 since the composition of continuous linear maps is separately continuous.

When $u_j \in \mathcal{D}'_{\pi_j}(M_j)$, $j = 1, 2$, both fibered products $u_1 \underset{\pi_j}{\times} u_2$, $j = 1, 2$ makes sense. Starting with (31) and applying Fubini Theorem for distributions, we get their equality and this also allows to take into account the extra transversality assumptions (29) in order to conclude, by the previous method, that $u_1 \underset{\pi_2}{\times} u_2$ is transversal with respect to $\rho \circ \mathrm{pr}_1, \pi$ and $\sigma \circ \mathrm{pr}_2$ and depends continuously on u_1 and u_2. $\qquad \square$

Consider a commutative diagram

$$(36)$$

where f is a C^∞ map and π, ρ are submersions. If $u \in \mathcal{E}'(M, \Omega_M)$, the push-forward of u by f is given by $\langle f_* u, g \rangle = \langle u, g \circ f \rangle$ and if moreover u is transversal with respect to π, then $f_* u$ is given by the C^∞ family $((f|_{M_b})_* u_b)$, $b \in B$. We obtain a map

$$(37) \qquad f_* : \mathcal{E}'_\pi(M, \Omega_M) \longrightarrow \mathcal{E}'_\rho(N, \Omega_N).$$

Since f is not necessarily proper, we can not extend f_* to $\mathcal{D}'_\pi$, nevertheless:

Proposition 18. *Let $\varphi \in C^\infty(M)$ such that $f : \mathrm{supp}(\varphi) \longrightarrow N$ is proper. Then the map*

$$(38) \qquad \mathcal{D}'_\pi(M, \Omega_M) \longrightarrow \mathcal{D}'_\rho(N, \Omega_N)$$

$$(39) \qquad u \longmapsto f_*(\varphi u)$$

is well defined and continuous.

Proof. Under the assumption on the support of φ, we easily get that $g \longmapsto \varphi.g \circ f$ maps continuously $C^\infty_{\mathrm{fc}-\rho}(N)$ into $C^\infty_{\mathrm{fc}-\pi}(M)$. The result follows. $\qquad \square$

3. CONVOLUTION OF TRANSVERSAL DISTRIBUTIONS ON GROUPOIDS

We apply these observations in the context of Lie groupoids.

A Lie groupoid is a manifold G endowed with the additional following structures:

- two surjective submersions $r, s : G \rightrightarrows G^{(0)}$ onto a manifold $G^{(0)}$ called the space of units.
- An embedding $u : G^{(0)} \longrightarrow G$, which allows to consider $G^{(0)}$ as a submanifold of G and then such that

$$(40) \qquad r(x) = x \quad , \quad s(x) = x, \quad \text{for all } x \in G^{(0)}.$$

- A C^∞ map

$$(41) \qquad i : G \longrightarrow G, \quad \gamma \longmapsto \gamma^{-1}$$

  called inversion and satisfying $s(\gamma^{-1}) = r(\gamma)$ and $r(\gamma^{-1}) = s(\gamma)$ for any γ.
- a C^∞ map

$$(42) \qquad m : G^{(2)} = \{(\gamma_1, \gamma_2) \in G^2 \; ; \; s(\gamma_1) = r(\gamma_2)\} \longrightarrow G, \quad (\gamma_1, \gamma_2) \longmapsto \gamma_1 \gamma_2$$

  called the multiplication, satisfying the relations, whenever they make sense

$$(43) \qquad (\gamma_1 \gamma_2)\gamma_3 = \gamma_1(\gamma_2 \gamma_3) \qquad r(\gamma)\gamma = \gamma \qquad \gamma s(\gamma) = \gamma$$

$$(44) \qquad \gamma \gamma^{-1} = r(\gamma) \qquad \gamma^{-1}\gamma = s(\gamma) \qquad r(\gamma_1 \gamma_2) = r(\gamma_1), \; s(\gamma_1 \gamma_2) = s(\gamma_2).$$

It follows from these axioms that i is a diffeomorphism equal to its inverse, m is a surjective submersion and γ^{-1} is the unique inverse of γ, for any γ, that is the only element of G satisfying $\gamma \gamma^{-1} = r(\gamma)$, $\gamma^{-1}\gamma = s(\gamma)$. These assertions need a proof, and the unfamiliar reader is invited to consult for instance [13] and references therein.

It is customary to write

$$G_x = s^{-1}(x), \quad G^x = r^{-1}(x), \; G^y_x = G_x \cap G^y, \; m_x = m|_{G^x \times G_x} : G^x \times G_x \longrightarrow G.$$

G_x, G^x are submanifolds and G^x_x is a Lie group. The submersion $d : (\gamma_1, \gamma_2) \mapsto \gamma_1 \gamma_2^{-1}$ defined on $G \overset{s}{\times} G$ is called division of G.

Obviously, Lie groups, C^∞ vector bundles, principal bundles, are Lie groupoids. Also, for any manifold X, the manifold $X \times X$ inherits a canonical structure of Lie groupoid with unit space X

and multiplication given by $(x,y).(y,z) = (x,z)$. The reader can find in [25, 19, 3, 18, 16, 5, 15, 17, 6] more concrete examples.

The Lie algebroid $A(G)$ of a Lie groupoid G is the fiber bundle $TG|_{G^{(0)}}/TG^{(0)}$ over $G^{(0)}$. It can be identified with Ker $ds|_{G^{(0)}}$ or Ker $dr|_{G^{(0)}}$. Its dual $A^*(G)$ is the conormal bundle of $G^{(0)}$.

We recall the construction of the canonical convolution algebra $C_c^\infty(G, \Omega^{1/2})$ [3, 7] associated with any Lie groupoid G. The product of convolution

$$(45) \qquad C_c^\infty(G, \Omega^{1/2}) \times C_c^\infty(G, \Omega^{1/2}) \xrightarrow{\ *\ } C_c^\infty(G, \Omega^{1/2})$$

is given by the integral

$$(46) \qquad f * g(\gamma) = \int_{\gamma_1\gamma_2=\gamma} f(\gamma_1)g(\gamma_2), \quad \gamma \in G$$

which is well defined and gives an internal operation as soon as we take

$$(47) \qquad \Omega^{1/2} = \Omega^{1/2}(\ker dr) \otimes \Omega^{1/2}(\ker ds) = \Omega^{1/2}(\ker dr \oplus \ker ds).$$

To understand this point, we recall

Lemma 19. [3, 7]. *Denoting by m the multiplication map of G and by $\mathrm{pr}_1, \mathrm{pr}_2 : G \times G \to G$ the natural projection maps, we have a canonical isomorphism*

$$(48) \qquad \mathrm{pr}_1^*(\Omega^{1/2}) \otimes \mathrm{pr}_2^*(\Omega^{1/2})|_{G^{(2)}} \simeq \Omega(\ker dm) \otimes m^*(\Omega^{1/2}).$$

Proof. We note $G_\gamma^{(2)}$ the fiber of m at γ, that is $G_\gamma^{(2)} = \{(\gamma_1, \gamma_2) \in G^{(2)} \ ; \ \gamma_1\gamma_2 = \gamma\}$.

Now, the restricted map $\mathrm{pr}_1 : G_\gamma^{(2)} \to G^{r(\gamma)}$ being a diffeomorphism, we have a canonical isomorphism of vector bundles

$$TG_\gamma^{(2)} \xrightarrow{\simeq} \mathrm{pr}_1^*(TG^{r(\gamma)}) = \mathrm{pr}_1^*(\ker dr)|_{G_\gamma^{(2)}}, \ \ (\gamma_1, \gamma_2, X_1, X_2) \longmapsto (\gamma_1, \gamma_2, X_1).$$

Similarly, $TG_\gamma^{(2)} \simeq \mathrm{pr}_2^*(TG_{s(\gamma)})$. Moreover the map

$$\mathrm{pr}_1^*(\ker ds)|_{G_\gamma^{(2)}} \xrightarrow{\simeq} G_\gamma^{(2)} \times T_\gamma G_{s(\gamma)}, \ \ (\gamma_1, \gamma_2, X_1) \longmapsto (\gamma_1, \gamma_2, (dR_{\gamma_2})_{\gamma_1}(X_1))$$

provides a canonical trivialisation of the vector bundle $\mathrm{pr}_1^*(\ker ds)|_{G_\gamma^{(2)}}$. The same holds for

$$\mathrm{pr}_2^*(\ker dr)|_{G_\gamma^{(2)}} \simeq G_\gamma^{(2)} \times T_\gamma G^{r(\gamma)}.$$

With these isomorphisms in hand, we get

$$\begin{aligned}
\mathrm{pr}_1^*(\Omega^{1/2})|_{G_\gamma^{(2)}} \otimes \mathrm{pr}_2^*(\Omega^{1/2})|_{G_\gamma^{(2)}} &\simeq \Omega^{1/2}(\mathrm{pr}_1^*(\ker dr \oplus \ker ds)|_{G_\gamma^{(2)}} \oplus \mathrm{pr}_2^*(\ker dr \oplus \ker ds)|_{G_\gamma^{(2)}}) \\
&\simeq \Omega^{1/2}(TG_\gamma^{(2)} \oplus T_\gamma G_{s(\gamma)} \oplus T_\gamma G^{r(\gamma)} \oplus TG_\gamma^{(2)}) \\
&\simeq \Omega(TG_\gamma^{(2)}) \otimes \Omega^{1/2}(T_\gamma G_{s(\gamma)} \oplus T_\gamma G^{r(\gamma)}).
\end{aligned}$$

This gives the canonical isomorphim (48). $\qquad\qquad\square$

Since in the basic formula (46) the function under sign of integration

$$G_\gamma^{(2)} \ni (\gamma_1, \gamma_2) \mapsto f(\gamma_1)g(\gamma_2) \in \left(\mathrm{pr}_1^*(\Omega^{1/2}) \otimes \mathrm{pr}_2^*(\Omega^{1/2})\right)_{(\gamma_1,\gamma_2)}$$

is a C^∞ section of the bundle $\left(\mathrm{pr}_1^*(\Omega^{1/2}) \otimes \mathrm{pr}_2^*(\Omega^{1/2})\right)|_{G_\gamma^{(2)}}$, Lemma 19 shows that (46) is the integral of a one density, canonically associated with f, g over the submanifold $m^{-1}(\gamma)$ and that the result is a C^∞ section of $\Omega^{1/2}$. Further computations on densities show that the statement

$$(49) \qquad f * g(\gamma) = \int_{G^{r(\gamma)}} f(\gamma_1)g(\gamma_1^{-1}\gamma) = \int_{G_{s(\gamma)}} f(\gamma\gamma_2^{-1})g(\gamma_2)$$

makes sense and is true. The involution on $C_c^\infty(G, \Omega^{1/2})$ is also natural in terms of densities

$$f^\star = \overline{i^*(f)}, \quad f \in C_c^\infty(G, \Omega^{1/2})$$

where i is the induced vector bundle isomorphism over the inversion map of G

$$\ker dr \oplus \ker ds \longrightarrow \ker dr \oplus \ker ds, \ (\gamma, X_1, X_2) \longmapsto (\gamma^{-1}, di(X_2), di(X_1)).$$

The spaces $C^\infty(G, \Omega^{-1/2} \otimes \Omega_G)$ and $C_c^\infty(G, \Omega^{-1/2} \otimes \Omega_G)$ are endowed with their usual Fréchet and $\mathcal{LF}$ topological vector space structures and we denote by $\mathcal{E}'(G, \Omega^{1/2})$ and $\mathcal{D}'(G, \Omega^{1/2})$ their topological duals. The choice of densities is made so that we have canonical embeddings

$$C^\infty(G, \Omega^{1/2}) \hookrightarrow \mathcal{D}'(G, \Omega^{1/2}) \text{ and } C_c^\infty(G, \Omega^{1/2}) \hookrightarrow \mathcal{E}'(G, \Omega^{1/2}).$$

For simplicity, we assume in the sequel that $G^{(0)}$ is compact, thus $\mathcal{E}'(G, \Omega^{1/2}) = \mathcal{D}'_{\mathrm{fc}-\pi}(G, \Omega^{1/2})$ and $\mathcal{E}'_\pi(G, \Omega^{1/2}) = \mathcal{P}'_\pi(G, \Omega^{1/2})$ if $\pi \in \{r, s\}$.

Theorem 20. *The bilinear map*

$$(50) \qquad \qquad \mathcal{E}'_s(G, \Omega^{1/2}) \times \mathcal{E}'(G, \Omega^{1/2}) \overset{*}{\longrightarrow} \mathcal{E}'(G, \Omega^{1/2})$$

$$(u, v) \longmapsto u * v = m_*(u \underset{s}{\times} v)$$

is well defined and separately continuous. Also, the maps

$$(51) \qquad \mathcal{D}'(G, \Omega^{1/2}) \overset{*}{\longrightarrow} \mathcal{D}'(G, \Omega^{1/2}) \qquad \qquad and \qquad \mathcal{D}'_s(G, \Omega^{1/2}) \overset{*}{\longrightarrow} \mathcal{D}'_s(G, \Omega^{1/2})$$

$$v \longmapsto u_0 * v = m_*(u_0 \underset{s}{\times} v) \qquad \qquad \qquad u \longmapsto u * v_0 = m_*(u \underset{s}{\times} v_0)$$

are well defined and continuous for any $u_0 \in \mathcal{E}'_s(G, \Omega^{1/2})$ and $v_0 \in \mathcal{E}'(G, \Omega^{1/2})$. Similar statements are available for r-transversal distributions used as right variables. We get by restriction separately continuous bilinear maps

$$(52) \qquad \qquad \mathcal{E}'_\pi(G, \Omega^{1/2}) \times \mathcal{E}'_\pi(G, \Omega^{1/2}) \overset{*}{\longrightarrow} \mathcal{E}'_\pi(G, \Omega^{1/2})$$

for $\pi = r$ and $\pi = s$. The space $(\mathcal{E}'_\pi(G, \Omega^{1/2}), *)$ is an associative algebra with unit given by

$$(53) \qquad \qquad \langle \delta, f \rangle = \int_{G^{(0)}} f, \quad f \in C^\infty(G, \Omega^{-1/2} \otimes \Omega_G).$$

In particular $(\mathcal{E}'_{r,s}(G, \Omega^{1/2}), *)$ is an associative unital algebra with involution given by

$$(54) \qquad \qquad u^\star = \overline{i^*(u)}.$$

Proof. Applying Proposition 16 to the case $M_1 = M_2 = G$, $B = G^{(0)}$, $\pi_1 = s$, $\pi_2 = r$ and $\sigma : G \to \{\mathrm{pt}\}$, one gets a distribution $u \underset{s}{\times} v \in \mathcal{D}'(G^{(2)}, \Omega^{1/2})$ which depends continuously on u and v. Since $u \in \mathcal{E}'$ on can choose $\phi \in C_c^\infty(G)$ such that $u = \phi u$. Then

$$u \underset{s}{\times} v = \varphi u \underset{s}{\times} v$$

where $\varphi = \phi \circ \mathrm{pr}_1 |_{G^{(2)}}$ and Proposition 18 can be applied to the case $f = m$ with $B = \{\mathrm{pt}\}$. This gives that $u * v$ is well defined for $v \in \mathcal{D}'$ and the continuity of $v \mapsto u * v$ on $\mathcal{E}', \mathcal{D}'$ as well. For fixed $v \in \mathcal{E}'$, one gets the continuity of $u \mapsto u * v$ on $\mathcal{E}'_s, \mathcal{D}'_s$ in the same way.

To prove the statement involving (52) for $\pi = s$ we apply Proposition 16 to $M_1 = M_2 = G$, $B = G^{(0)}$, $\pi_1 = s$, $\pi_2 = r$ and $\sigma = s$ and Proposition 18 to $\rho = s$ and $\pi = s \circ \mathrm{pr}_2$.

The associativity of $*$ on distributions follows by continuity and density of $C_c^\infty(G, \Omega^{1/2})$.

We check that the integral defining δ has an intrinsic meaning and gives a unit in $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$. Since $TG|_{G^{(0)}} = TG^{(0)} \oplus \ker ds$, we have $\Omega(G)|_{G^{(0)}} = \Omega(G^{(0)}) \otimes \Omega(\ker ds)$. On the other hand the inversion gives a canonical isomorphism between the bundles $\ker ds|_{G^{(0)}}$ and $\ker dr|_{G^{(0)}}$, thus

$$\Omega(\ker ds) = \Omega^{1/2}(\ker ds) \otimes \Omega^{1/2}(\ker ds) \simeq \Omega^{1/2}(\ker ds) \otimes \Omega^{1/2}(\ker dr) = \Omega^{1/2}$$

Through these canonical identifications, any $f \in C^\infty(G, \Omega^{-1/2} \otimes \Omega_G)$ gives by restriction to $G^{(0)}$ a one density on $G^{(0)}$, which gives a well defined meaning to $\delta(f)$. Obviously

$$r_*(f\delta) = s_*(f\delta) = f|_{G^{(0)}} \in C^\infty(G^{(0)}) \subset \mathcal{D}'(G^{(0)}), \text{ for any } f \in C^\infty(G),$$

in particular $\delta \in \mathcal{E}'_{r,s}(G, \Omega^{1/2})$. If $\delta^x \in \mathcal{D}'(G^x)$, $x \in G^{(0)}$ is the associated C^∞ family, we then get by Remark 6

$$\langle \delta^x, \phi \rangle = r_*(\delta\widetilde{\phi})(x) = \phi(x), \quad \text{for any } \phi \in C_c^\infty(G^x) \text{ and } \widetilde{\phi} \in C_c^\infty(G) \text{ such that } \widetilde{\phi}|_{G^x} = \phi.$$

It follows that for any $f \in C^\infty(G, \Omega^{-1/2} \otimes \Omega_G)$,

$$\langle u * \delta, f \rangle = \int_{x \in G^{(0)}} \langle u_x \otimes \delta^x, (f \circ m)|_{G_x \times G^x} \rangle = \int_{x \in G^{(0)}} \langle u_x, f|_{G_x} \rangle = \langle u, f \rangle.$$

The proof of the equality $\delta * u = u$ is similar. The assertion about the involution is obvious. □

In particular, when one of the two factors is in C_c^∞, the convolution product is defined without any restriction on the other factor. We give a sufficient condition for the result to be C^∞.

Proposition 21. *The convolution product gives by restriction a bilinear separetely continuous map*

$$\mathcal{D}'_r(G, \Omega^{1/2}) \times C_c^\infty(G, \Omega^{1/2}) \xrightarrow{*} C^\infty(G, \Omega^{1/2}).$$

The analogous statement with C^∞ functions on the left and s-transversal distributions on the right also holds. The map $u \mapsto u * \cdot$ mapping $\mathcal{D}'_r(G, \Omega^{1/2})$ to $\mathcal{L}(C_c^\infty(G, \Omega^{1/2}), C^\infty(G, \Omega^{1/2}))$ is injective.

Proof. If $u = (u^y)_y \in \mathcal{D}'_r$, the map

(55) $$\gamma \mapsto \langle u^{r(\gamma)}(\cdot), f((\cdot)^{-1}\gamma) \rangle$$

is C^∞ and by definition of the convolution product we get

$$\langle u * f, \phi \rangle = \int_{\gamma_2 \in G} \langle u^{r(\gamma_2)}(\cdot), f((\cdot)^{-1}\gamma_2) \rangle \phi(\gamma_2).$$

Thus $u * f$ coincides with the C^∞ function (55). The continuity of $u \mapsto u * f$ is given by Theorem 20 and repeating the argument given in its proof, one gets the continuity of $f \mapsto u * f$ on $C_0^\infty(K, \Omega^{1/2}) = \{ f \in C^\infty \; ; \; \text{supp}(f) \subset K \}$ for any compact $K \subset G$. The results follows by inductive limit.

Now, the vanishing of $u * f$ for any f and the previous expression for $u * f$ shows that $u^x = 0$, for any x, and thus $u = 0$. □

Remark 22. Note that if in the previous proposition we suppose that u has compact support $K \subset G$, then $u * f$ can be defined for any map $f \in C^\infty(G, \Omega^{1/2})$. Moreover for any $f \in C_c^\infty(G, \Omega^{1/2})$, then $u * f$ is also compactly supported and $\text{supp}(u * f) \subset K.\text{supp}(f)$.

4. G-OPERATORS

We recall the notion of G-operators given in [16] and we add a notion of adjoint for them.

Definition 3. *A (left) G-operator is a continuous linear map $P : C_c^\infty(G, \Omega^{1/2}) \to C^\infty(G, \Omega^{1/2})$ such that there exists a family $P_x : C_c^\infty(G_x, \Omega_{G_x}^{1/2}) \longrightarrow C^\infty(G_x, \Omega_{G_x}^{1/2})$, $x \in G^{(0)}$ of operators such that*

$$(56) \qquad P(f)|_{G_x} = P_x(f|_{G_x}), \ \forall f \in C_c^\infty(G, \Omega^{1/2}), \ \forall x \in G^{(0)}$$

$$(57) \qquad P_{r(\gamma)} \circ R_\gamma = R_\gamma \circ P_{s(\gamma)}, \ \forall \gamma \in G.$$

A G-operator P is said adjointable if there exists a G-operator Q such that

$$(58) \qquad (P(f)|g) = (f|Q(g)) ; \quad f, g \in C_c^\infty(G, \Omega^{1/2}).$$

Here $(f|g) = f^\star * g$ is the $C_c^\infty(G, \Omega^{1/2})$-valued pre-hilbertian product of $C_c^\infty(G, \Omega^{1/2})$.

We note Op_G and $\mathrm{Op}_G^\star$ respectively the linear spaces of G-operators and adjointable ones.

We say that G-operator P is supported in K if $\mathrm{supp}(P(f)) \subset K.\mathrm{supp}(f)$ for all f. The subspaces of compactly supported G-operators are denoted $\mathrm{Op}_{G,c}, \mathrm{Op}_{G,c}^\star$.

Looking at $C_c^\infty(G, \Omega^{1/2})$ and $C^\infty(G, \Omega^{1/2})$ as right $C_c^\infty(G, \Omega^{1/2})$-modules for the convolution product, G-operators can be characterized in a simple way.

Proposition 23. *A linear operator $P : C_c^\infty(G, \Omega^{1/2}) \to C^\infty(G, \Omega^{1/2})$ is a G-operator if and only if it is continuous and*

$$P(f * g) = P(f) * g \quad \forall f, g \in C_c^\infty(G, \Omega^{1/2}).$$

In other words, $\mathrm{Op}_G = \mathcal{L}_{C_c^\infty(G, \Omega^{1/2})}(C_c^\infty(G, \Omega^{1/2}), C^\infty(G, \Omega^{1/2}))$.

Proof. Let $P \in \mathrm{Op}_G$. Let us write p_x for the Schwartz kernel of P_x. For any f, g compactly supported and $\gamma \in G$

$$
\begin{aligned}
P(f * g)(\gamma) &= \int_{\gamma_2 \in G_{s(\gamma)}} \int_{\gamma_1 \in G_{s(\gamma)}} p_{s(\gamma)}(\gamma, \gamma_2) f(\gamma_2 \gamma_1^{-1}) g(\gamma_1) \\
&= \int_{\gamma_1 \in G_{s(\gamma)}} \left(\int_{\gamma_2 \in G_{s(\gamma)}} p_{s(\gamma)}(\gamma, \gamma_2)(R_{\gamma_1^{-1}} f)(\gamma_2) \right) g(\gamma_1) \\
&= \int_{\gamma_1 \in G_{s(\gamma)}} \left(\int_{\gamma_2 \in G_{s(\gamma)}} p_{r(\gamma_1)}(\gamma \gamma_1^{-1}, \gamma_2) f(\gamma_2) \right) g(\gamma_1) \\
&= \int_{\gamma_1 \in G_{s(\gamma)}} P(f)(\gamma \gamma_1^{-1}) g(\gamma_1) = P(f) * g(\gamma).
\end{aligned}
$$

Conversely, let $f \in C_c^\infty(G, \Omega^{1/2})$ and $x \in G^{(0)}$ such that $f|_{G_x} = 0$. observe that $(g * f)|_{G_x} = 0$ for any $g \in C_c^\infty(G, \Omega^{1/2})$. It follows that $P(g * f)|_{G_x} = P(g) * f|_{G_x} = 0$. Choose a sequence $\phi_n \in C_c^\infty(G, \Omega^{1/2})$ converging to δ in $\mathcal{E}_r'$. Then $\phi_n * f$ converges to f in $C_c^\infty(G, \Omega^{1/2})$ and therefore

$$P(f)(\gamma) = \lim P(\phi_n * f)(\gamma) = 0 \quad \forall \gamma \in G_x.$$

In other words, $P(f)|_{G_x}$ only depends on $f|_{G_x}$ and we can define P_x for any x by

$$P_x(f) = P(\widetilde{f})|_{G_x} \quad \forall f \in C_c^\infty(G_x, \Omega_{G_x}^{1/2}) \text{ and } \widetilde{f} \in C_c^\infty(G, \Omega^{1/2}) \text{ such that } \widetilde{f}|_{G_x} = f.$$

Let $\gamma \in G_x^y$. Then for any $\gamma' \in G^y$ and $f \in C_c^\infty(G, \Omega^{1/2})$, we have

$$
\begin{aligned}
R_\gamma(P_x(\phi_n * f))(\gamma') &= P(\phi_n * f)(\gamma'\gamma) \\
&= P(\phi_n) * f(\gamma'\gamma) = P(\phi_n) * (R_\gamma f)(\gamma') = P(\phi_n * (R_\gamma f))(\gamma').
\end{aligned}
$$

Taking the limit in this equality gives (57). $\qquad\square$

Let $u \in \mathcal{D}_r'(G, \Omega^{1/2})$. Using Propositions 21 and 23, we can define $P \in \mathrm{Op}_G$ by setting $P(f) = u * f$ for any $f \in C_c^\infty(G, \Omega^{1/2})$.

Conversely, let $P \in \mathrm{Op}_G$ and $p_x \in \mathcal{D}'(G_x \times G_x)$ the Schwartz kernel of P_x, $x \in G^{(0)}$. Since

$$
\gamma \longmapsto P(f)(\gamma) = \int p_{s(\gamma)}(\gamma, \gamma_1) f(\gamma_1)
$$

is C^∞ for any f, we get that $\gamma \mapsto p_{s(\gamma)}(\gamma, \cdot)$ belongs to $\mathcal{D}'_{\mathrm{pr}_1}(G \times G)$ and then using Proposition 12, it restricts to the map $G^{(0)} \ni x \mapsto p_x(x, \cdot)$ belonging to $\mathcal{D}'_s(G)$. Defining $k_P \in \mathcal{D}'_r(G)$ by $k_P(\gamma) = p_{r(\gamma)}(r(\gamma), \gamma^{-1})$, we get for any $f \in C_c^\infty(G, \Omega^{1/2})$, $x, y \in G^{(0)}$ and $\gamma \in G_x^y$

$$
\begin{aligned}
P(f)(\gamma) &= \int_{G_x} p_x(\gamma, \gamma_1) f(\gamma_1) = \int_{G_y} p_y(y, \gamma_1) f(\gamma_1 \gamma) \\
(59) \qquad\qquad &= \int_{G^y} p_y(y, \gamma_1^{-1}) f(\gamma_1^{-1}\gamma) = \langle (k_P)_y, f((\cdot)^{-1}\gamma) \rangle_{G^y} = k_P * f(\gamma).
\end{aligned}
$$

Thus P the operator given by left convolution with k_P. We call k_P *the convolution distributional kernel* of P. Note that $\mathrm{supp}(P) = \mathrm{supp}(k_P)$. We have proved

Theorem 24. *The map $P \mapsto k_P$ gives the isomorphisms*

$$
(60) \qquad\qquad \mathrm{Op}_G \simeq \mathcal{D}_r'(G, \Omega^{1/2}) \quad\text{and}\quad \mathrm{Op}_{G,c} \simeq \mathcal{E}_r'(G, \Omega^{1/2}).
$$

If $k_P \in \mathcal{D}_{r,s}'(G, \Omega^{1/2})$ then P is obviously adjointable and $k_{P^\star} = (k_P)^\star$. Conversely, if P as an adjoint Q then

$$
(61) \qquad\qquad (k_P * f)^\star * g = (f^\star * k_P^\star) * g = f^\star * (k_Q * g) ; \quad f, g \in C_c^\infty(G, \Omega^{1/2}),
$$

hence $k_P^\star = k_Q \in \mathcal{D}_s'(G, \Omega^{1/2}) \cap \mathcal{D}_r'(G, \Omega^{1/2})$. Thus Theorem 24 yields

Corollary 25. *The map $P \to k_P$ gives an isomorphism*

$$
(62) \qquad\qquad \mathrm{Op}_G^\star \simeq \mathcal{D}_{r,s}'(G, \Omega^{1/2}).
$$

Remark 26. Rephrasing the previous results, we have, for instance

$$
\mathrm{Op}_G \simeq \mathcal{L}_s(C_c^\infty(G, \Omega^{1/2}), C^\infty(G^{(0)})).
$$

where we have replaced $\mathcal{L}_{C^\infty(G^{(0)})}$ by $\mathcal{L}_s$ to emphasize that the $C^\infty(G^{(0)})$-module structure on $C_c^\infty(G, \Omega^{1/2})$ is given by s. Also

$$
\mathrm{Op}_G^\star \simeq \mathcal{L}_{r,s}(C_c^\infty(G, \Omega^{1/2}), C^\infty(G^{(0)})).
$$

where $\mathcal{L}_{r,s} = \mathcal{L}_s \cap \mathcal{L}_r$. In terms of Schwartz kernel theorems for submersions, G-operators thus appear as "semi-regular" distributions (see Treves [21, p.532]) since, for $\pi = s$ or $\pi = r$

$$
\mathcal{D}'(G, \Omega^{1/2}) \simeq \mathcal{L}_\pi(C_c^\infty(G, \Omega^{1/2}), \mathcal{D}'(G^{(0)})).
$$

Now observe that if $k_P \in \mathcal{E}_{r,s}'(G, \Omega^{1/2})$, Theorem 20 implies that P extends continuously to a map $\mathcal{D}'(G, \Omega^{1/2}) \longrightarrow \mathcal{D}'(G, \Omega^{1/2})$ sending the subspace $\mathcal{E}_{r,s}'$ to $\mathcal{E}_{r,s}'$. This leads to another characterization of adjointness.

Proposition 27. *A compactly supported G-operator P is adjointable if and only if it extends continuously to a map*

$$\widetilde{P} : \mathcal{D}'(G, \Omega^{1/2}) \longrightarrow \mathcal{D}'(G, \Omega^{1/2})$$

such that $\widetilde{P}(\delta) \in \mathcal{D}'_{r,s}(G, \Omega^{1/2})$. In that case, $\widetilde{P} = k_P * \cdot$.

Proof. Let $u \in \mathcal{D}'(G, \Omega^{1/2})$ and $(u_n) \subset C_c^\infty(G, \Omega^{1/2})$ a sequence converging to u in $\mathcal{D}'$. We have

$$\widetilde{P}(u * f) = \lim P(u_n * f) = \lim P(u_n) * f = \widetilde{P}(u) * f, \quad \forall f \in C_c^\infty(G, \Omega^{1/2}).$$

Thus $\widetilde{P}$ is automatically $C_c^\infty(G, \Omega^{1/2})$-right linear. It follows that

$$k_P * f = P(f) = P(\delta * f) = \widetilde{P}(\delta) * f, \quad \forall f \in C_c^\infty(G, \Omega^{1/2})$$

which proves that $k_P = \widetilde{P}(\delta) \in \mathcal{D}'_{r,s}(G, \Omega^{1/2})$ and that $\widetilde{P}$ is given by left convolution with k_P. $\square$

5. Convolution on groupoids and wave front sets

We now turn to some microlocal aspects of the convolution of distributions on groupoids. In view of Proposition 7, it is natural to call r-transversal any (conic) subset $W \subset T^*G \setminus 0$ such that $W \cap \ker dr^\perp = \emptyset$, indeed in that case

$$(63) \qquad \mathcal{D}'_W(G, \Omega^{1/2}) \subset \mathcal{D}'_r(G, \Omega^{1/2}).$$

Similarly, W is called s-transversal if $W \cap \ker ds^\perp = \emptyset$ and we call bi-transversal any set which is both r and s-transversal. We then introduce

$$(64) \qquad \mathcal{D}'_a(G, \Omega^{1/2}) = \{u \in \mathcal{D}'(G, \Omega^{1/2}) \; ; \; \text{WF}(u) \text{ is bi-transversal}\}$$

and $\mathcal{E}'_a = \mathcal{D}'_a \cap \mathcal{E}'$. We call them *admissible* distributions. From Proposition 7, we get

$$(65) \qquad \mathcal{D}'_a(G, \Omega^{1/2}) \subset \mathcal{D}'_{r,s}(G, \Omega^{1/2}).$$

Example 1. Observe that $A^*G \setminus 0$ is bi-transversal. Since $\Psi(G) = I(G, G^{(0)}) \subset \mathcal{D}'_{A^*G}(G)$ (see [14]) we get

$$(66) \qquad \Psi(G) \subset \mathcal{D}'_a(G, \Omega^{1/2}).$$

Theorem 20 and Proposition 21 can be reused in various ways for subspaces of distributions with transversal wave front sets. We only record the main one: the convolution product restricts to a bilinear map

$$(67) \qquad \mathcal{E}'_a(G, \Omega^{1/2}) \times \mathcal{E}'_a(G, \Omega^{1/2}) \xrightarrow{\;*\;} \mathcal{E}'_{r,s}(G, \Omega^{1/2}),$$

and we strengthten this result as follows, by using the cotangent groupoid structure of Coste-Dazord-Weinstein (see Appendix).

Theorem 28. *For any $u_1, u_2 \in \mathcal{E}'_a(G, \Omega^{1/2})$, we have $u_1 * u_2 \in \mathcal{E}'_a(G, \Omega^{1/2})$ and*

$$(68) \qquad \text{WF}(u_1 * u_2) \subset \text{WF}(u_1) * \text{WF}(u_2)$$

where on the right, $*$ denotes the product of the symplectic groupoid $T^*G \rightrightarrows A^*G$. In particular $(\mathcal{E}'_a(G, \Omega^{1/2}), *)$ is a unital involutive subalgebra of $(\mathcal{E}'_{r,s}(G, \Omega^{1/2}), *)$.

Proof. Let $u_j \in \mathcal{E}'_a(G, \Omega^{1/2})$ and set $W_j = \mathrm{WF}(u_j)$, $j = 1, 2$. We first show that the fibered product $u_1 \underset{\pi}{\times} u_2$ (where $\pi = r, s$ indifferently) given by Proposition 16, coincides with the distribution obtained by the functorial operations in [10, Theorems 8.2.9, 8.2.4]:

$$(69) \qquad u_1 \underset{\pi}{\times} u_2 = \rho^*(u_1 \otimes u_2) \in \mathcal{D}'(G^{(2)}, \Omega(\ker dm) \otimes m^*(\Omega^{1/2})),$$

where $\rho : G^{(2)} \hookrightarrow G^2$. By [10, Theorem 8.2.9]), we know that

$$(70) \qquad \mathrm{WF}(u_1 \otimes u_2) \subset W_1 \times W_2 \cup W_1 \times (G \times \{0\}) \cup (G \times \{0\}) \times W_2,$$

and to apply [10, Theorems 8.2.4], we just need to check that

$$(71) \qquad \mathrm{WF}(u_1 \otimes u_2) \cap N^* G^{(2)} = \emptyset.$$

Observe that $N^* G^{(2)} = \ker m_\Gamma \subset \Gamma^{(2)}$ and $\ker ds^\perp = \ker r_\Gamma$. Thus, if

$$\delta_j = (\gamma_j, \xi_j) \in T^*_{\gamma_j} G \text{ and } (\delta_1, \delta_2) \in \mathrm{WF}(u_1 \otimes u_2) \cap N^* G^{(2)}$$

then $(\delta_1, \delta_2) \in \Gamma^{(2)}$ and

$$(72) \qquad r_\Gamma(\delta_1) = r_\Gamma(\delta_1 \delta_2) = (r(\gamma_1), 0).$$

By the *s*-transversality assumption on W_1 and the relation (70), this implies $\delta_1 = (\gamma_1, 0)$ and $\delta_2 \in W_2$. On the other hand

$$(73) \qquad s_\Gamma(\delta_2) = s_\Gamma(\delta_1 \delta_2) = (s(\gamma_2), 0),$$

which contradicts the *r*-transversality of W_2, and this proves (71). Therefore, the right hand side in (69) is well defined by [10, Theorems 8.2.4] and it coincides with the left hand side, which is obvious after pairing with test functions. Now

$$(74) \qquad u_1 * u_2 = m_*(u_1 \underset{\pi}{\times} u_2) = m_* \rho^*(u_1 \otimes u_2)$$

and thus, using [10, Theorems 8.2.4] and [8, (3.6), p. 328],

$$(75) \qquad \mathrm{WF}(u_1 * u_2) \subset m_* \rho^* \, \mathrm{WF}(u_1 \otimes u_2) \,.$$

Here $\rho^* : T^* G^2 \longrightarrow T^* G^{(2)}$ is the restriction of linear forms and, for any $\widetilde{W} \subset T^* G^{(2)}$,

$$m_*(\widetilde{W}) = \{(\gamma, \xi) \in T^* G \; ; \; \exists (\gamma_1, \gamma_2) \in m^{-1}(\gamma), \; (\gamma_1, \gamma_2, {}^t dm_{\gamma_1, \gamma_2}(\xi)) \in \widetilde{W} \cup G^{(2)} \times 0\}.$$

Since m is submersive, ${}^t dm_{\gamma_1, \gamma_2}$ is injective and the term $G^{(2)} \times 0$ can be removed. By definition of the multiplication of $\Gamma = T^* G$, we get, for any $W \subset T^* G^2$, the equivalence

$$(76) \qquad \gamma_1 \gamma_2 = \gamma \text{ and } (\gamma_1, \gamma_2, {}^t dm_{\gamma_1, \gamma_2}(\xi)) \in \rho^*(W) \; \Leftrightarrow \exists (\delta_1, \delta_2) \in \Gamma^{(2)} \cap W, \; \delta_1 \delta_2 = (\gamma, \xi).$$

Thus,

$$(77) \qquad m_* \rho^* W = m_\Gamma(W \cap \Gamma^{(2)}).$$

By *r*-transversality of $\mathrm{WF}(u_1)$, we have $s_\Gamma(\mathrm{WF}(u_1)) \subset A^* G \setminus 0$, so $\mathrm{WF}(u_1) \times (G \times \{0\}) \cap \Gamma^{(2)} = \emptyset$. Similarly, *s*-transversality of $\mathrm{WF}(u_2)$ gives $(G \times \{0\}) \times \mathrm{WF}(u_2) \cap \Gamma^{(2)} = \emptyset$. It follows that $\mathrm{WF}(u_1 \otimes u_2) \cap \Gamma^{(2)} = (\mathrm{WF}(u_1) \times \mathrm{WF}(u_2)) \cap \Gamma^{(2)}$ and therefore

$$m_\Gamma(\mathrm{WF}(u_1 \otimes u_2) \cap \Gamma^{(2)}) = m_\Gamma((\mathrm{WF}(u_1) \times \mathrm{WF}(u_2)) \cap \Gamma^{(2)}) = \mathrm{WF}(u_1) * \mathrm{WF}(u_2)$$

which proves (68). Clearly, $W_1 * W_2$ is s or r-transversal if the same holds respectively for W_1 and W_2, so (68) implies $u_1 * u_2 \in \mathcal{E}'_a$, therefore $\mathcal{E}'_a$ is a subalgebra of $\mathcal{E}'_{r,s}$.

Finally, since $\mathrm{WF}(\delta) = A^* G \setminus 0$, we have $\delta \in \mathcal{E}'_a$ and since $\mathrm{WF}(u^\star) = i_\Gamma(\mathrm{WF}(u))$, we conclude that $\mathcal{E}'_a$ is unital and involutive. $\qquad \square$

Looking at the proof of the Theorem, we see that the assumptions on $\mathrm{WF}(u_j)$ can be significanlty relaxed in order to conserve the property (71) and then to be able to define the convolution product $u_1 * u_2$ by the right hand side of (74).

Firstly, if $W \subset T^*G \setminus 0$, then $W \times (G \times 0) \cap \ker m_\Gamma = \emptyset$. Indeed, if $(\gamma_1, \xi_1, \gamma_2, 0) \in W \times (G \times \{0\}) \cap \Gamma^{(2)}$, we can choose $t_1 \in T_{\gamma_1} G$ such that $\xi_1(t_1) \neq 0$ since $\xi_1 \neq 0$ by assumption. Using a local section β of r such that $\beta(s(\gamma_1)) = \gamma_2$ and setting $t_2 = d\beta ds(t_1) \in T_{\gamma_2} G$, we get $(t_1, t_2) \in T_{(\gamma_1, \gamma_2)} G^{(2)}$ and $\xi_1(t_1) + 0(t_2) \neq 0$, that is $\xi_1 \oplus 0 \neq 0$ which proves that $(\gamma_1, \xi_1, \gamma_2, 0) \notin \ker m_\Gamma$.

Arguing identically on $(G \times 0) \times W$ we get the equivalence, for any distributions u_1, u_2

(78) $$\mathrm{WF}(u_1 \otimes u_2) \cap \ker m_\Gamma = \emptyset \Leftrightarrow \mathrm{WF}(u_1) \times \mathrm{WF}(u_2) \cap \ker m_\Gamma = \emptyset.$$

This is again the condition (71) which is sufficient to define $\rho^*(u_1 \otimes u_2) = u_1 \otimes u_2|_{G^{(2)}}$ and there the convolution product under additional suitable supports conditions.

Theorem 29. *Let $W_j \subset T^*G \setminus 0$ be closed cones such that*

(79) $$W_1 \times W_2 \cap \ker m_\Gamma = \emptyset$$

and set $W_1 \overline{\ast} W_2 = m_\Gamma((W_1 \times W_2 \cup W_1 \times 0 \cup 0 \times W_2) \cap \Gamma^{(2)})$. Then the map

(80) $$\mathcal{E}'_{W_1}(G, \Omega^{1/2}) \times \mathcal{E}'_{W_2}(G, \Omega^{1/2}) \xrightarrow{\;*\;} \mathcal{E}'_{W_1 \overline{\ast} W_2}(G, \Omega^{1/2})$$

(81) $$(u_1, u_2) \longmapsto m_*(u_1 \otimes u_2|_{G^{(2)}})$$

is separately sequentially continuous and coincides with the convolution product on $C_c^\infty(G, \Omega^{1/2})$.

Proof. Under the assumption made on W_1, W_2, we can apply [10, Theorems 8.2.4, 8.2.9] to find that the bilinear map

(82) $$\mathcal{D}'_{W_1}(G, \Omega^{1/2}) \times \mathcal{D}'_{W_2}(G, \Omega^{1/2}) \longrightarrow \mathcal{D}'_{\rho^*(W_1 \bar{\times} W_2)}(G^{(2)}, \Omega^{1/2})$$

$$(u_1, u_2) \longmapsto u_1 \otimes u_2|_{G^{(2)}}$$

is well defined, sequentially separately continuous for the natural notion of convergence of sequences in the spaces $\mathcal{D}'_W$ [10, 8], and also separately continuous for the normal topology of these spaces [2]. Above, we have set for convenience $W_1 \bar{\times} W_2 = W_1 \times W_2 \cup W_1 \times 0 \cup 0 \times W_2$.

To apply m_* and get a continuous map for the same topologies, we restrict ourselves to compactly supported distributions and we get

(83) $$\mathcal{E}'_{W_1}(G, \Omega^{1/2}) \times \mathcal{E}'_{W_2}(G, \Omega^{1/2}) \xrightarrow{(\cdot \otimes \cdot)|_{G^{(2)}}} \mathcal{E}'_{\rho^*(W_1 \bar{\times} W_2)}(G^{(2)}, \Omega^{1/2}) \xrightarrow{m_*} \mathcal{E}'_{W_1 \overline{\ast} W_2}(G, \Omega^{1/2}).$$

Indeed, the formulas (75) and (77) are still valid here and give the last distribution space above. $\square$

If u_1 or u_2 is smooth then $\mathrm{WF}(u_1) \times \mathrm{WF}(u_2)$ is empty and (79) is trivially satisfied, thus

Corollary 30. *The convolution product of Theorem 29 gives by restriction the maps*

(84) $$\mathcal{E}'(G, \Omega^{1/2}) \times C_c^\infty(G, \Omega^{1/2}) \xrightarrow{\;*\;} \mathcal{E}'_{s_\Gamma^{-1}(0)}(G, \Omega^{1/2}),$$

(85) $$C_c^\infty(G, \Omega^{1/2}) \times \mathcal{E}'(G, \Omega^{1/2}) \xrightarrow{\;*\;} \mathcal{E}'_{r_\Gamma^{-1}(0)}(G, \Omega^{1/2}).$$

As we said, bi-transversal subsets of $T^*G \setminus 0$ satisfy (79). Actually,

Corollary 31. *Let W_1, W_2 be any subsets of $T^*G \setminus 0$. If W_1 is s-transversal (resp. W_2 is r-transversal) then the assumption (79) is satisfied and $W_1 * W_2$ is s-transversal (resp. W_2 r-transversal) .*

Proof. Use the equalities $s_\Gamma \circ m_\Gamma = s_\Gamma \circ \mathrm{pr}_2$ and $r_\Gamma \circ m_\Gamma = r_\Gamma \circ \mathrm{pr}_1$. $\square$

Remark 32. Theorems 20 and 29 do not apply exactly to the same situations. For instance, consider the pair groupoid $G = \mathbb{R} \times \mathbb{R}$. On one hand, using the relation $\ker m_\Gamma = ((\ker ds)^\perp \times (\ker dr)^\perp) \cap (T^*G)^{(2)}$ and Remark 9, it is easy to obtain pairs of distributions $(u_1, u_2) \in \mathcal{E}'_s(\mathbb{R}^2) \times \mathcal{E}'(\mathbb{R}^2)$ for which only Theorem 20 can be applied to define $u_1 * u_2$. On the other hand, consider the distributions $u_1 = \delta_{(0,0)}$ and $u_2 = \delta_{(1,1)}$, whose wave fronts are respectively $W_1 = \{(0, 0, \xi, \eta) \ ; \ (\xi, \eta) \neq (0, 0)\}$ and $W_2 = \{(1, 1, \xi, \eta) \ ; \ (\xi, \eta) \neq (0, 0)\}$. These distributions are neither s nor r transversal, but $W_1 \times W_2 \cap \Gamma^{(2)} = \emptyset$, hence the convolution $u_1 * u_2$ on G can only be defined by Theorem 29 (note that $u_1 * u_2 = 0$; less peculiar examples can be easily constructed).

Of course, both convolution products coincide when both make sense, since the equality (69) is valid as soon as $(\mathrm{WF}(u_1) \times \mathrm{WF}(u_2)) \cap \ker m_\Gamma = \emptyset$.

6. Appendix : The cotangent groupoid of Coste-Dazord-Weinstein

We recall the definition of the cotangent groupoid of Coste-Dazord-Weinstein. We explain the construction of the source and target map given in [4] and we enlighten the role played by the differential of the multiplication map of G. This is a pedestrian approach based on concrete differential geometry while more conceptual developments can be found in [20, 13].

Let G be a Lie groupoid whose multiplication is denoted by m, source and target by s, r and inversion by i. Differentiating all the structure maps of G, we get that $TG \rightrightarrows TG^{(0)}$ is a Lie groupoid whose multiplication is given by dm, source and target by ds, dr and inversion by di. Hence, it is natural to try to transpose everything to get a groupoid structure on $\Gamma = T^*G$. Following this idea, it is natural to decide that the product $(\gamma_1, \xi_1).(\gamma_2, \xi_2) \in T^*G$ of two elements $(\gamma_j, \xi_j) \in T^*G$ is defined by $(\gamma_1\gamma_2, \xi)$ where ξ is the solution of the equation

$$(86) \qquad {}^t dm_{(\gamma_1, \gamma_2)}(\xi) = (\xi_1, \xi_2)|_{T_{(\gamma_1, \gamma_2)}G^{(2)}}.$$

Indeed, $m : G^{(2)} \longrightarrow G$ being a submersion, ${}^t dm_{(\gamma_1, \gamma_2)}$ is injective for all $(\gamma_1, \gamma_2) \in G^{(2)}$ and ξ, when it exists, is therefore unique. In that case, we have

$$(87) \qquad \xi = {}^t dm^{-1}_{(\gamma_1, \gamma_2)} \rho(\xi_1, \xi_2)$$

where $\rho : T^*_{G^{(2)}} G^2 \longrightarrow T^*G^{(2)}$ is the restriction of linear forms and we introduce the notations

$$(88) \qquad \xi = \xi_1 \oplus \xi_2 \text{ and } m_\Gamma(\gamma_1, \xi_1, \gamma_2, \xi_2) = (\gamma_1\gamma_2, \xi_1 \oplus \xi_2).$$

The equation (86) has a solution ξ if and only if

$$(89) \qquad (\xi_1, \xi_2) \in \mathrm{Im}\, {}^t dm_{(\gamma_1, \gamma_2)}.$$

Since $\mathrm{Im}\, {}^t dm_{(\gamma_1, \gamma_2)} = (\ker dm_{(\gamma_1, \gamma_2)})^\perp$, this is equivalent to

$$(90) \qquad \xi_1(t_1) + \xi_2(t_2) = 0, \quad \forall(t_1, t_2) \in \ker dm_{(\gamma_1, \gamma_2)}.$$

Let us explicit $\ker dm \subset TG^{(2)}$. Let

$$L_\gamma : G^{s(\gamma)} \longrightarrow G^{r(\gamma)}, \gamma' \mapsto \gamma\gamma' \text{ and } R_\gamma : G_{r(\gamma)} \longrightarrow G_{s(\gamma)}, \gamma' \mapsto \gamma'\gamma$$

be the left and right multiplication maps of G. Let $(\gamma_1, \gamma_2) \in G^{(2)}$ and set $\gamma = \gamma_1\gamma_2$, $x = s(\gamma_1)$. Parametrizing $G^{(2)}_\gamma = m^{-1}(\gamma)$ by $G^{r(\gamma)} \ni \eta \mapsto (\eta, \eta^{-1}\gamma)$, we find, after a routine computation:

$$(91) \qquad (t_1, t_2) \in \ker dm_{(\gamma_1, \gamma_2)} \Leftrightarrow t_1 = dL_{\gamma_1} di(t), \ t_2 = dR_{\gamma_2}(t), \text{ for some } t \in T_x G_x.$$

It follows that (89) is equivalent to the equality

$$(92) \qquad {}^t dR_{\gamma_2}(\xi_2) = -{}^t d(L_{\gamma_1} \circ i)(\xi_1) \in (T_x G_x)^*,$$

where it is understood that R_{γ_2} and $L_{\gamma_1} \circ i$ are differentiated at $\gamma = x$ and that the linear forms ξ_1, ξ_2 are restricted to the ranges of the corresponding differential maps. The same abuse of notations is used below without further notice. We then define elements $\overline{s}(\xi_1), \overline{r}(\xi_2)$ belonging to $A_x^* G = (T_x G/T_x G^{(0)})^*$ by

$$(93) \qquad \overline{s}(\xi_1)(t + u) = {}^t dL_{\gamma_1}(\xi_1)(t) \text{ for all } t + u \in T_x G^x \oplus T_x G^{(0)} = T_x G,$$

$$(94) \qquad \overline{r}(\xi_2)(t + u) = {}^t dR_{\gamma_2}(\xi_2)(t) \text{ for all } t + u \in T_x G_x \oplus T_x G^{(0)} = T_x G.$$

Differentiating the relation $\gamma^{-1}\gamma = s(\gamma)$ at $\gamma = x$ we get the relation

$$(95) \qquad di + \mathrm{id} = ds + dr$$

which yields $-di(t) \equiv t \mod T_x G^{(0)}$, $\forall t \in T_x G$. Thus, (92), and then (89), is equivalent to

$$(96) \qquad \overline{r}(\xi_2) = \overline{s}(\xi_1) \in A_x^* G.$$

This leads to the definitions

$$(97) \qquad s_\Gamma(\gamma, \xi) = (s(\gamma), \overline{s}(\xi)) \in A^* G \text{ and } r_\Gamma(\gamma, \xi) = (r(\gamma), \overline{r}(\xi)) \in A^* G, \quad \forall(\gamma, \xi) \in T^* G.$$

Finally, we denote $u_\Gamma : A^* G \hookrightarrow T^* G$ the canonical inclusion and we set

$$(98) \qquad i_\Gamma(\gamma, \xi) = (\gamma^{-1}, -({}^t di_\gamma)^{-1}(\xi)), \quad \forall(\gamma, \xi) \in T^* G.$$

Theorem 33. [4]. *Let G be a Lie groupoid. The space $\Gamma = T^* G$ is a Lie groupoid with unit space $A^* G$ and structural maps given by $s_\Gamma, r_\Gamma, m_\Gamma, i_\Gamma$ and u_Γ (respectively, source, target, multiplication, inversion and inclusion of unit maps).*

Remark 34. (1) The Lie algebroid of G is sometimes defined by $AG = \ker ds|_{G^{(0)}}$. In that picture, we deduce from (92) that s_Γ and r_Γ have to be defined by replacing $\overline{s}, \overline{r}$ by

$$(99) \qquad \widetilde{s}(\xi) = -{}^t d(L_\gamma \circ i)(\xi) \qquad \text{and} \qquad \widetilde{r}(\xi) = {}^t dR_\gamma(\xi).$$

(2) The submanifold $\Gamma^{(2)}$ of composable pairs in Γ is given by

$$(100) \qquad \Gamma^{(2)} = \{(\delta_1, \delta_2) \in T^*_{G^{(2)}} G^2 \ ; \ \rho(\delta_1, \delta_2) \in (\ker dm)^\perp\}$$

and $m_\Gamma = {}^t dm^{-1} \circ \rho$.

(3) The graph of m_Γ is canonically isomorphic to the conormal space of the graph of m:

$$(101) \qquad \mathrm{Gr}(m_\Gamma) \ni (\gamma, \xi, \gamma_1, \xi_1, \gamma_2, \xi_2) \longrightarrow (\gamma, -\xi, \gamma_1, \xi_1, \gamma_2, \xi_2) \in N^* \mathrm{Gr}(m).$$

Since $N^* \mathrm{Gr}(m)$ is Lagrangian in $T^* G \times T^* G \times T^* G$, we get that $\mathrm{Gr}(m_\Gamma)$ is Lagrangian in $(-T^* G) \times T^* G \times T^* G$, that is, Γ is a symplectic groupoid.

Finally, we remember that $T^* G$ is also a vector bundle over G, and we note $p : T^* G \to G$ the projection map. The following result is useful and obvious from the construction detailed above.

Proposition 35.   (1) *The subspace of composable pairs $\Gamma^{(2)}$ is a vector bundle over $G^{(2)}$ and $m_\Gamma : \Gamma^{(2)} \to \Gamma$ is a vector bundle homomorphism:*

(102)
$$\begin{array}{ccc}
\Gamma^{(2)} & \xrightarrow{\ m_\Gamma\ } & \Gamma \\
\downarrow{\scriptstyle (p,p)} & & \downarrow{\scriptstyle p} \\
G^{(2)} & \xrightarrow{\ m\ } & G
\end{array}$$

whose kernel is the conormal space of $G^{(2)}$ into G^2: $\ker m_\Gamma = N^*G^{(2)}$.

(2) *The maps $r_\Gamma, s_\Gamma : \Gamma \to A^*G$ are also vector bundle homomorphisms:*

(103)
$$\begin{array}{ccc}
\Gamma & \xrightarrow{\ s_\Gamma\ } & A^*G \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
G & \xrightarrow{\ s\ } & G^{(0)}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\Gamma & \xrightarrow{\ r_\Gamma\ } & A^*G \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
G & \xrightarrow{\ r\ } & G^{(0)}
\end{array}$$

and $\ker r_\Gamma = (\ker ds)^\perp$, $\ker s_\Gamma = (\ker dr)^\perp$.

We finish this review with two basic examples, the first one being the historical one [4].

Example 2. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. We have immediately*

(104)
$$s_\Gamma(g,\xi) = L_g^*\xi \in \mathfrak{g}^* \text{ and } r_\Gamma(g,\xi) = R_g^*\xi \in \mathfrak{g}^*.$$

When $s_\Gamma(g_1,\xi_1) = r_\Gamma(g_2,\xi_2)$, we get $(g_1,\xi_1)(g_2,\xi_2) = (g_1g_2,\xi)$ with ξ characterized by:

(105)
$$\xi(dm_{(g_1,g_2)}(t_1,t_2)) = \xi_1(t_1) + \xi_2(t_2).$$

Since $dm_{(g_1,g_2)}(t_1,t_2) = dR_{g_2}(t_1) + dL_{g_1}(t_2)$, we obtain $\xi = R^*_{g_2^{-1}}\xi_1 = L^*_{g_1^{-1}}\xi_2$. Thus

(106)
$$(g_1,\xi_1)(g_2,\xi_2) = (g_1g_2, R^*_{g_2^{-1}}\xi_1) \text{ when } L^*_{g_1}\xi_1 = R^*_{g_2}\xi_2.$$

On the other hand, we recall that G acts on $\mathfrak{g}^*$ by

(107)
$$Ad_g^*.\xi = L_g^*R^*_{g^{-1}}\xi.$$

This gives rise to the transformation groupoid $G \ltimes \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*$ whose source, target, multiplication and inversion are thus given by

(108) $s(g,\xi) = Ad_g^*.\xi, \quad r(g,\xi) = \xi, \quad (g_1,\xi_1)(g_2, Ad_{g_1}^*.\xi_1) = (g_1g_2,\xi_1), \quad (g,\xi)^{-1} = (g^{-1}, Ad_g^*.\xi).$

Now, the vector bundle trivialization $\Phi : T^*G \longrightarrow G \times \mathfrak{g}^*$, $(g,\xi) \longmapsto (g, R_g^*\xi)$, gives a Lie groupoid isomorphism $\Phi : T^*G \longrightarrow G \ltimes \mathfrak{g}^*$. For instance, we check

$$\begin{aligned}
\Phi((g_1,\xi_1)(g_2,\xi_2)) &= \Phi(g_1g_2, R^*_{g_2^{-1}}\xi_1) = (g_1g_2, R^*_{g_1g_2}R^*_{g_2^{-1}}\xi_1) = (g_1g_2, R^*_{g_1}\xi_1) \\
&= (g_1, R^*_{g_1}\xi_1).(g_2, R^*_{g_2}\xi_2) \quad \text{since } Ad_{g_1}^*.R^*_{g_1}\xi_1 = L^*_{g_1}\xi_1 = R^*_{g_2}\xi_2 \\
&= \Phi(g_1,\xi_1).\Phi(g_2,\xi_2).
\end{aligned}$$

Example 3. *We take $G = X \times X \times Z \rightrightarrows X \times Z$ (cartesian product of the pair groupoid $X \times X$ with the space Z). Here we have*

$$\Gamma^{(0)} = A^*G = \{(x,x,z,\xi,-\xi,0) \ ; \ (x,\xi) \in T^*X, z \in Z\}.$$

Let $\gamma = (x,y,z)$ and $\xi = (\zeta,\eta,\sigma) \in T_\gamma^*G$. Then $\overline{s}(\xi) \in T^*_{(y,y,z)}X \times X \times Z$ is given by $\eta \in T_y^*X \simeq 0 \times T_y^*X \times 0$ after extension by 0 onto the subspace of vectors of the form (u,u,w). This is similar for $\overline{r}(\xi) \in T^*_{(x,x,z)}X \times X \times Z$, starting with $\zeta \in T_x^*X \simeq T_x^*X \times 0 \times 0$. Using

$$(u,v,w) = (u-v,0,0) + (v,v,w) = (0,v-u,0) + (u,u,w),$$

we get $s_\Gamma(x, y, z, \xi, \eta, \sigma) = (y, y, z, -\eta, \eta, 0)$, $r_\Gamma(x, y, z, \xi, \eta, \sigma) = (x, x, z, \xi, -\xi, 0)$ *and*

$$(109) \qquad (x, y, z, \xi, \eta, \sigma).(y, x', z, -\eta, \xi', \sigma') = (x, x', z, \xi, \xi', \sigma + \sigma').$$

Note that if $Z = \{\text{pt}\}$, $\Gamma = T^*(X \times X)$ *is isomorphic to the pair groupoid* $T^*X \times T^*X$, *with isomorphism given by* $T^*(X \times X) \longrightarrow T^*X \times T^*X$; $(x, y, \zeta, \eta) \mapsto (x, \zeta, y, -\eta)$.

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